

Contact geometric description of distributed-parameter port-Hamiltonian systems with respect to Stokes-Dirac structures and its information geometry

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Abstract

This paper studies distributed-parameter systems on Riemannian manifolds with respect to Stokes-Dirac structures in a language of contact geometry with fiber bundles. For the class where energy functionals are quadratic, it is shown that distributed-parameter port-Hamiltonian systems with respect to Stokes-Dirac structures on one, two, and three dimensional Riemannian manifolds are written in terms of contact Hamiltonian vector fields on bundles. Their fiber spaces are contact manifolds and base spaces are Riemannian manifolds. In addition, for a class of distributed-parameter port-Hamiltonian systems, information geometry induced from contact manifolds and convex energy functionals is introduced and briefly discussed.

1 Introduction

Contact geometry and symplectic one are often referred to as twins, and then they have been studied from viewpoints of pure mathematics[1] and applied mathematics. Applications of contact geometry are found in science and foundation of engineering. They include equilibrium thermodynamics[2, 3, 4], nonequilibrium thermodynamics[5, 6], statistical mechanics[7, 8, 9], fluid mechanics[10], electromagnetism[11], control theory[12], non-conservative system[13, 14] and so on. Also, some models for LC circuits in contact with a thermal environment can be written in terms of contact geometry[15], and it was shown that a LC circuit model without any external source and Maxwell's equations in media without source can be formulated in terms of contact geometry[16, 17]. In Ref. [17], Maxwell's equations without source can be written in terms of a bundle whose fiber space is a contact manifold. In Refs. [6, 17] some links between information geometry and contact geometry were clarified. Here information geometry is a geometrization of parametric statistics[18]. With information geometry, dynamics on so-called statistical manifolds was discussed [19, 20] and applications on optimization problems were studied[21].

Dirac structures on manifolds are some sub-bundles known to be extensions of symplectic and Poisson manifolds[22], and they have been intensively studied from various viewpoints. These structures allow to describe constraint mechanical systems and interconnected systems as implicit Hamiltonian systems[23, 24, 25]. Engineering applications of these include electric circuit theory[24]. For example, LC circuit models can be viewed as a degenerate Lagrangian system and those can be analyzed by the use of a Dirac structure[26]. It was then shown that Dirac structures on manifolds can be extended so that distributed-parameter systems with boundary energy flow can be described[27]. This extended structure is a sub-bundle and is referred to as the Stokes-Dirac structure. As examples of such, it was demonstrated that Maxwell's equations, telegraph equations, vibrating strings, and ideal fluids are formulated in terms of Stokes-Dirac structures[27]. In addition, there are further extensions from the original Stokes-Dirac structures[28, 29] and it is expected that various sophisticated approaches to distributed-parameter systems will be provided.

In this paper a partial overlap between contact geometry and the theory of Stokes-Dirac structures is shown. Since Maxwell's equations have been formulated on a bundle whose fiber space is a contact

manifold[17] and on a Stokes-Dirac structure[27] as mentioned above, it can be expected that there is a link between these two geometric descriptions. We thus focus on this link by generalizing the work of Ref. [17]. To this end, Maxwell's equations are treated as a distributed-parameter system with respect to a Stokes-Dirac structure in this paper. Then it will be shown how some class of distributed-parameter systems with respect to Stokes-Dirac structures are written as contact Hamiltonian vector fields. In addition, by generalizing Ref. [17], information geometry for a class of distributed-parameter port-Hamiltonian systems will be introduced and then discussed.

2 Preliminaries

In this section a brief summary of several geometries is given. This summary is used throughout this paper in order to describe various statements in the following sections.

2.1 Mathematical symbols, definitions, and known theorems

Throughout this paper, geometric objects are assumed smooth. A point on an n -dimensional manifold $\xi \in \mathcal{M}$ is often identified with a set of values of local coordinates $x(\xi) = \{x^1(\xi), \dots, x^n(\xi)\}$.

The following definition of bundle and the definitions of related objects are used in this paper. More mathematically rigorous definitions can be found in Ref.[30] and so on.

Definition 2.1. (*Bundle or fiber bundle*): Let \mathcal{B} be a $d_{\mathcal{B}}$ -dimensional manifold with local coordinates $\zeta = \{\zeta_1, \dots, \zeta_{d_{\mathcal{B}}}\}$, \mathcal{F} a $d_{\mathcal{F}}$ -dimensional manifold, \mathcal{M} a $(d_{\mathcal{B}} + d_{\mathcal{F}})$ -dimensional manifold, $\pi : \mathcal{M} \rightarrow \mathcal{B}$ a projection, G a group acting on \mathcal{F} , and $\{U_i\}$ an open covering of \mathcal{B} with $\phi_i : U_i \times \mathcal{F} \rightarrow \pi^{-1}(U_i)$ such that $\pi\phi_i(\zeta, u) = \zeta$. Then the set $(\mathcal{M}, \pi, \mathcal{B})$ or $(\mathcal{M}, \pi, \mathcal{B}, \mathcal{F}, G)$ is referred to as a bundle or a fiber bundle, \mathcal{B} a base space, \mathcal{F} a fiber space, G a structure group, ϕ_i a local trivialization, and \mathcal{M} a total space. Furthermore, let $t_{ij}(\zeta) := \phi_{i,\zeta}^{-1} \circ \phi_{j,\zeta}$ be an element of G ($t_{ij} : U_i \cap U_j \rightarrow G$), where $\phi_{i,\zeta}(u) = \phi_i(\zeta, u)$ for $U_i \cap U_j \neq \emptyset$. Then $\{t_{ij}\}$ are referred to as transition functions.

Definition 2.2. (*Trivial bundle and non-trivial bundle*): If a transition function for a bundle can be chosen to be identical, then the bundle is referred to as a trivial bundle. Otherwise, the bundle is referred to as a non-trivial bundle.

Non-identical transition functions are used for describing non-trivial bundles. For example, the Möbius band can be constructed with this formulation[30]. In this paper trivial bundles are only considered.

A special class of sub-space of a bundle is considered in this paper, and the definition is given as follows.

Definition 2.3. (*Sub-bundle*): Let $(\mathcal{M}, \pi, \mathcal{B})$ and $(\mathcal{M}', \pi', \mathcal{B})$ be bundles. If the two conditions,

1. \mathcal{M}' is a submanifold of \mathcal{M} ,
2. $\pi' = \pi|_{\mathcal{M}'}$

are satisfied, then $(\mathcal{M}', \pi', \mathcal{B})$ is referred to as a sub-bundle of $(\mathcal{M}, \pi, \mathcal{B})$.

Definition 2.4. (*Section*): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle, and $f : \mathcal{B} \rightarrow \mathcal{M}$ a map such that $\pi \circ f = \text{Id}_{\mathcal{B}}$. Then f is referred to as a section. Here $\text{Id}_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B}$ denotes the identity map on \mathcal{B} . The space of sections is denoted as $\Gamma\mathcal{M}$.

A set of vector fields on a manifold \mathcal{M} is denoted $\Gamma T\mathcal{M}$, the tangent space at $\xi \in \mathcal{M}$ as $T_{\xi}\mathcal{M}$, the cotangent space at $\xi \in \mathcal{M}$ as $T_{\xi}^*\mathcal{M}$, a set of q -form fields $\Gamma\Lambda^q\mathcal{M}$ with $q \in \{0, \dots, \dim\mathcal{M}\}$, and a set of tensor fields $\Gamma T_q^{q'}\mathcal{M}$ with $q, q' \in \{0, 1, \dots\}$. To express tensor fields the direct product is denoted \otimes . Einstein notation, when an index variables appear twice in a single term it implies summation of all the values of the index, is used. The Kronecker delta is denoted $\delta_a^b, \delta_{ab}, \dots$, and its value is unity when $a = b$, and zero when $a \neq b$. The exterior derivative acting on $\Gamma\Lambda^q\mathcal{M}$ is denoted $d : \Gamma\Lambda^q\mathcal{M} \rightarrow \Gamma\Lambda^{q+1}\mathcal{M}$, and the interior product operator with $X \in \Gamma T\mathcal{M}$ as $\iota_X : \Gamma\Lambda^q\mathcal{M} \rightarrow \Gamma\Lambda^{q-1}\mathcal{M}$. Given a map Φ between two manifolds, the pull-back is denoted Φ^* , and the push-forward Φ_* . Then one can define the Lie derivative

acting on tensor fields with respect to $X \in \Gamma T\mathcal{M}$, which is denoted $\mathcal{L}_X : \Gamma T_q^{q'}\mathcal{M} \rightarrow \Gamma T_q^{q'}\mathcal{M}$. It follows that $\mathcal{L}_X\beta = (\iota_X d + d\iota_X)\beta$, for any $\beta \in \Gamma\Lambda^q\mathcal{M}$, which is referred to as the Cartan formula. One can also define a derivative along a given vector field X , called the covariant derivative, denoted $\nabla_X : \Gamma T_q^0\mathcal{M} \rightarrow \Gamma T_q^0\mathcal{M}$. The action is explicitly given by specifying the connection coefficients Γ_{ab}^c , ($a, b, c \in \{1, \dots, n\}$) with $n = \dim\mathcal{M}$ such that $\nabla_{X_a} X_b = \Gamma_{ab}^c X_c$ where $\{X_1, \dots, X_n\} \in \Gamma T\mathcal{M}$ is a basis. For a given $S \in \Gamma T_q^0\mathcal{M}$, an object $\nabla S \in \Gamma T_{q+1}^0\mathcal{M}$ is defined such that $(\nabla S)(X, Y_1, \dots, Y_q) = (\nabla_X S)(Y_1, \dots, Y_q)$, where $X, Y_1, \dots, Y_q \in \Gamma T\mathcal{M}$. For example it can be shown that $\nabla f = df$ for $f \in \Gamma T_0^0\mathcal{M}$, and that $(\nabla df)(Y, \partial/\partial x^a) = Y^b(\partial^2 f/\partial x^a \partial x^b - \Gamma_{ba}^c \partial f/\partial x^c)$, where $Y = Y^b \partial/\partial x^b \in \Gamma T\mathcal{M}$.

2.2 Geometry of fiber bundles

Given a bundle $(\mathcal{M}, \pi, \mathcal{B})$, the space of q -forms on \mathcal{B} and the space of q' -forms on \mathcal{M} can be introduced as follows.

Definition 2.5. (*Horizontal form*): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle with $\dim \mathcal{B} = d_{\mathcal{B}}$ and $\dim \mathcal{M} = d_{\mathcal{B}} + d_{\mathcal{F}}$, ζ coordinates for \mathcal{B} with $\zeta = \{\zeta^1, \dots, \zeta^{d_{\mathcal{B}}}\}$, (ζ, u) coordinates for \mathcal{M} with $u = \{u^1, \dots, u^{d_{\mathcal{F}}}\}$, and $\{\alpha_{i_1 \dots i_q}\} \in \Gamma\Lambda^0\mathcal{M}$ some functions. A q -form on the bundle $(\mathcal{M}, \pi, \mathcal{B})$ of the form

$$\alpha_{\mathbb{H}} = \alpha_{i_1 \dots i_q}(\zeta, u) d\zeta^{i_1} \wedge \dots \wedge d\zeta^{i_q},$$

is referred to as a horizontal q -form. The space of horizontal q -forms is denoted as $\Gamma\Lambda_{\mathbb{H}}^q\mathcal{M}$.

Definition 2.6. (*Vertical form*): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle, $\dim \mathcal{B} = d_{\mathcal{B}}$ and $\dim \mathcal{M} = d_{\mathcal{B}} + d_{\mathcal{F}}$, ζ a set of coordinates for \mathcal{B} , (ζ, u) a set of coordinates for \mathcal{M} with $\zeta = \{\zeta^1, \dots, \zeta^{d_{\mathcal{B}}}\}$ and $u = \{u^1, \dots, u^{d_{\mathcal{F}}}\}$, and $\{\alpha_{i_1 \dots i_q}\} \in \Gamma\Lambda^0\mathcal{M}$ some functions. A q -form on the bundle $(\mathcal{M}, \pi, \mathcal{B})$ of the form

$$\alpha_{\mathbb{V}} = \alpha_{i_1 \dots i_q}(\zeta, u) du^{i_1} \wedge \dots \wedge du^{i_q},$$

is referred to as a vertical q -form. The space of vertical q -forms is denoted as $\Gamma\Lambda_{\mathbb{V}}^q\mathcal{M}$. In addition, vertical 0-forms are referred to as vertical functions.

The wedge product of a horizontal q -form and a vertical q' -form can be defined. Then one defines the following.

Definition 2.7. (*Mixed form*): If a $(q + q')$ -form $\alpha_{\mathbb{M}} \in \Gamma\Lambda^{q+q'}\mathcal{M}$ can be written as

$$\alpha_{\mathbb{M}} = \beta_{\mathbb{H}} \wedge \gamma_{\mathbb{V}},$$

with some $\beta_{\mathbb{H}} \in \Gamma\Lambda_{\mathbb{H}}^q\mathcal{M}$ and $\gamma_{\mathbb{V}} \in \Gamma\Lambda_{\mathbb{V}}^{q'}\mathcal{M}$, then $\alpha_{\mathbb{M}}$ is referred to as a mixed (q, q') -form. The space of mixed (q, q') -forms on \mathcal{M} is denoted as $\Gamma\Lambda_{\mathbb{H}, \mathbb{V}}^{q, q'}\mathcal{M}$.

Definition 2.8. (*Vertical derivative*): Let $\alpha_{\mathbb{M}} \in \Gamma\Lambda_{\mathbb{H}, \mathbb{V}}^{q, q'}\mathcal{M}$ be a mixed (q, q') -form whose local expression can be written as

$$\alpha_{\mathbb{M}} = \alpha_{i_1 \dots i_q, j_1 \dots j_{q'}}(\zeta, u) d\zeta^{j_1} \wedge \dots \wedge d\zeta^{j_{q'}} \wedge du^{i_1} \wedge \dots \wedge du^{i_{q'}}.$$

The operator $d_{\mathbb{V}} : \Gamma\Lambda_{\mathbb{H}, \mathbb{V}}^{q, q'}\mathcal{M} \rightarrow \Gamma\Lambda_{\mathbb{H}, \mathbb{V}}^{q, q+1}\mathcal{M}$ whose action is such that

$$d_{\mathbb{V}}\alpha_{\mathbb{M}} = \frac{\partial \alpha_{i_1 \dots i_q, j_1 \dots j_{q'}}(\zeta, u)}{\partial u^{i_0}} du^{i_0} \wedge d\zeta^{i_1} \wedge \dots \wedge d\zeta^{i_q} \wedge du^{i_1} \wedge \dots \wedge du^{i_{q'}},$$

is referred to as the vertical derivative or the vertical exterior derivative.

Definition 2.9. (*Functional*): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle, $\alpha_{\mathbb{H}} \in \Gamma\Lambda_{\mathbb{H}}^q\mathcal{M}$ a horizontal q -form, $h \in \Gamma\Lambda_{\mathbb{V}}^0\mathcal{M}$ a vertical 0-form, and $\mathcal{B}_0 \subseteq \mathcal{B}$ a q -dimensional space. Then the integral over \mathcal{B}_0

$$\tilde{h}_{\mathcal{B}_0} = \int_{\mathcal{B}_0} h \alpha_{\mathbb{H}},$$

is referred to as a functional. The space of functionals is denoted as $\Gamma F\mathcal{M}$.

The functional derivative has been used in the infinite dimensional Hamiltonian systems[31], also in the theory of Stokes-Dirac structures[27]. In this paper this derivative is also used to describe distributed-parameter port-Hamiltonian systems.

Definition 2.10. (*Functional derivative*): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle, $\alpha_{\mathbb{M}}$ a mixed $(q, 0)$ -form on a q' -dimensional submanifold of \mathcal{B} , $\mathcal{B}_0 \subseteq \mathcal{B}$ a q' -dimensional subspace, $\tilde{h}_{\mathcal{B}_0}$ a functional depending on $\alpha_{\mathbb{M}}$, and $\eta \in \mathbb{R}$ a constant. Then the mixed $(q' - q, 0)$ -form $\delta \tilde{h}_{\mathcal{B}_0} / \delta \alpha_{\mathbb{M}} \in \Gamma \Lambda_{\mathbb{H}, \mathbb{V}}^{q' - q, 0} \mathcal{M}$ that is uniquely obtained by

$$\tilde{h}_{\mathcal{B}_0} [\alpha_{\mathbb{M}} + \eta \alpha'_{\mathbb{M}}] = \tilde{h}_{\mathcal{B}_0} [\alpha_{\mathbb{M}}] + \eta \int_{\mathcal{B}_0} \frac{\delta \tilde{h}_{\mathcal{B}_0}}{\delta \alpha_{\mathbb{M}}} \wedge \alpha'_{\mathbb{M}} + \mathcal{O}(\eta^2), \quad \forall \alpha'_{\mathbb{M}} \in \Gamma \Lambda_{\mathbb{H}, \mathbb{V}}^{q, 0} \mathcal{M}$$

is referred to as the functional derivative of $\tilde{h}_{\mathcal{B}_0}$ with respect to $\alpha_{\mathbb{M}}$.

Similar to the case of forms, let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle. Then the space of vector fields on \mathcal{B} and the space of vector fields on \mathcal{M} can be introduced as follows.

Definition 2.11. (*Horizontal vector field*): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle with $\dim \mathcal{B} = d_{\mathcal{B}}$ and $\dim \mathcal{M} = d_{\mathcal{B}} + d_{\mathcal{F}}$, ζ a set of coordinates for \mathcal{B} with $\zeta = \{\zeta^1, \dots, \zeta^{d_{\mathcal{B}}}\}$, (ζ, u) a set of coordinates for \mathcal{M} with $u = \{u^1, \dots, u^{d_{\mathcal{F}}}\}$, and $\{Y_1, \dots, Y_{d_{\mathcal{B}}}\} \in \Gamma \Lambda^0 \mathcal{M}$ some functions. A vector field on the bundle $(\mathcal{M}, \pi, \mathcal{B})$ of the form

$$Y_{\mathbb{H}} = Y_i(\zeta, u) \frac{\partial}{\partial \zeta^i},$$

is referred to as a horizontal vector field. The space of horizontal vector fields is denoted as $\Gamma T_{\mathbb{H}} \mathcal{M}$.

Remark 2.1. The dual of a horizontal vector field is a horizontal 1-form.

Definition 2.12. (*Vertical vector field*): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle with $\dim \mathcal{B} = d_{\mathcal{B}}$ and $\dim \mathcal{M} = d_{\mathcal{B}} + d_{\mathcal{F}}$, ζ a set of coordinates for \mathcal{B} with $\zeta = \{\zeta^1, \dots, \zeta^{d_{\mathcal{B}}}\}$, (ζ, u) a set of coordinates for \mathcal{M} with $u = \{u^1, \dots, u^{d_{\mathcal{F}}}\}$, and $\{Y_1, \dots, Y_{d_{\mathcal{F}}}\} \in \Gamma \Lambda^0 \mathcal{M}$ some functions. A vector field on the bundle $(\mathcal{M}, \pi, \mathcal{B})$ of the form

$$Y_{\mathbb{V}} = Y_i(\zeta, u) \frac{\partial}{\partial u^i},$$

is referred to as a vertical vector field. The space of vertical vector fields is denoted as $\Gamma T_{\mathbb{V}} \mathcal{M}$.

Remark 2.2. The dual of a vertical vector field is a vertical 1-form.

For a vertical q -form $\alpha_{\mathbb{V}}$, the action of the interior product with respect to a vertical vector field $Y_{\mathbb{V}}$ is denoted $\iota_{Y_{\mathbb{V}}} \alpha_{\mathbb{V}}$ and is similar to the action of a vector field Y to a q -form α , $\iota_Y \alpha$.

Definition 2.13. (*Interior product associated with a vertical vector field for vertical form*): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle with $\dim \mathcal{B} = d_{\mathcal{B}}$ and $\dim \mathcal{M} = d_{\mathcal{B}} + d_{\mathcal{F}}$, ζ a set of coordinates for \mathcal{B} with $\zeta = \{\zeta^1, \dots, \zeta^{d_{\mathcal{B}}}\}$, $\beta_{\mathbb{H}} \in \Gamma \Lambda_{\mathbb{H}}^q \mathcal{M}$ a horizontal q -form, $\gamma_{\mathbb{V}} \in \Gamma \Lambda_{\mathbb{V}}^{q'} \mathcal{M}$ a vertical q' -form, $Y_{\mathbb{V}} \in \Gamma T_{\mathbb{V}} \mathcal{M}$ a vertical vector field, and $\alpha_{\mathbb{M}} \in \Gamma \Lambda_{\mathbb{H}, \mathbb{V}}^{q, q'} \mathcal{M}$ a mixed (q, q') -form written as

$$\alpha_{\mathbb{M}} = \gamma_{\mathbb{V}} \wedge \beta_{\mathbb{H}}.$$

Then the action of $\iota_{Y_{\mathbb{V}}}$ to $\alpha_{\mathbb{M}}$, $\iota_{Y_{\mathbb{V}}} : \Gamma \Lambda_{\mathbb{H}, \mathbb{V}}^{q, q'} \mathcal{M} \rightarrow \Gamma \Lambda_{\mathbb{H}, \mathbb{V}}^{q, q' - 1} \mathcal{M}$ is defined as

$$\iota_{Y_{\mathbb{V}}} \alpha_{\mathbb{M}} = (\iota_{Y_{\mathbb{V}}} \gamma_{\mathbb{V}}) \wedge \beta_{\mathbb{H}}.$$

2.3 Riemannian geometry

In this subsection some of basics of Riemannian manifold are summarized. Roughly speaking, Riemannian geometry is a geometry where a class of metric tensor fields is provided for manifolds. In this paper Riemannian manifolds are used as base spaces of fiber bundles so that distributed-parameter port-Hamiltonian systems are geometrically described.

Definition 2.14. (*Riemannian manifold and Riemannian metric tensor field*): Let \mathcal{M} be an n -dimensional manifold and $g \in \Gamma T_2^0 \mathcal{M}$ a tensor field on \mathcal{M} . If g satisfies (i) $g|_\xi(X, Y) = g|_\xi(Y, X)$, (ii) $g|_\xi(X, X) \geq 0$, (iii) $g|_\xi(X, X) = 0$ iff $X = 0$, for $X, Y \in T_\xi \mathcal{M}$, then g is referred to as a Riemannian metric tensor field. An n -dimensional manifold together with a Riemannian metric tensor field g is denoted (\mathcal{M}, g) and this is referred to as an n -dimensional Riemannian manifold.

On an n -dimensional Riemannian manifold one can define various mathematical objects.

Definition 2.15. (*Orthonormal frame and orthonormal co-frame*): Let (\mathcal{M}, g) be an n -dimensional Riemannian manifold. If $\{X_1, \dots, X_n\} \in \Gamma T \mathcal{M}$ satisfies $g(X_a, X_b) = \delta_{ab}$, then $\{X_1, \dots, X_n\}$ is referred to as a g -orthonormal basis. In addition, if $\{\sigma^1, \dots, \sigma^n\} \in \Gamma \Lambda^1 \mathcal{M}$ satisfies $\sigma^a(X_b) = \delta_b^a$, then $\{\sigma^1, \dots, \sigma^n\}$ is referred to as a g -orthonormal co-frame.

Remark 2.3. One can express a Riemannian metric tensor field g in terms of a g -orthonormal co-frame $\{\sigma^a\}$ as $g = \delta_{ab} \sigma^a \otimes \sigma^b$.

Definition 2.16. (*Volume form*): Let \mathcal{M} be an n -dimensional manifold, and α an n -form. If α does not vanish at any point of \mathcal{M} , then α is referred to as a volume-form.

On Riemannian manifolds, a natural volume-form is given as below.

Definition 2.17. (*Canonical volume-form*): Let (\mathcal{M}, g) be an n -dimensional Riemannian manifold, and $\{\sigma^a\}$ its g -orthonormal co-frame. Then an n -form that does not vanish anywhere $\star 1 := \sigma^1 \wedge \dots \wedge \sigma^n$ is referred to as a canonical volume-form.

On Riemannian manifolds, the following map is induced.

Definition 2.18. (*Hodge map*): Let (\mathcal{M}, g) be an n -dimensional Riemannian manifold, and $\star 1$ its canonical volume-form. Define the map $\star : \Gamma \Lambda^q \mathcal{M} \rightarrow \Gamma \Lambda^{n-q} \mathcal{M}$ that satisfies

$$\star(\alpha \wedge \gamma) = \iota_Y \star \alpha, \quad \star(f \alpha) = f \star \alpha, \quad \star(\alpha + \beta) = (\star \alpha) + (\star \beta),$$

for all $\alpha, \beta \in \Gamma \Lambda^q \mathcal{M}$, ($q \in \{0, \dots, n\}$), $\gamma \in \Gamma \Lambda^1 \mathcal{M}$, $f \in \Gamma \Lambda^0 \mathcal{M}$, and Y is such that $\gamma = g(Y, -)$. Then, this map \star is referred to as the Hodge map.

The following formulae will be used.

Lemma 2.1. Let (\mathcal{Z}, g) be a 3-dimensional Riemannian manifold. Then it follows that

$$\star \star \alpha = \alpha,$$

for all $\alpha \in \Gamma \Lambda^q \mathcal{Z}$ with $q \in \{0, \dots, 3\}$. Thus, the inverse of \star denoted by \star^{-1} can be shown to be the same as \star :

$$\star^{-1} \alpha = \star \alpha,$$

for all q -form α .

Lemma 2.2. Let (\mathcal{Z}, g) be a 2-dimensional Riemannian manifold. Then it follows that

$$\star \star \alpha = (-1)^q \alpha,$$

for all $\alpha \in \Gamma \Lambda^q \mathcal{Z}$ with $q \in \{0, \dots, 2\}$. Thus, the inverse of \star denoted by \star^{-1} is written as

$$\star^{-1} \alpha = (-1)^q \star \alpha$$

for all q -form α .

Lemma 2.3. Let (\mathcal{Z}, g) be a 1-dimensional Riemannian manifold. Then it follows that

$$\star \star \alpha = \alpha,$$

for all $\alpha \in \Gamma \Lambda^q \mathcal{Z}$ with $q \in \{0, 1\}$. Thus, the inverse of \star denoted by \star^{-1} is the same as \star :

$$\star^{-1} \alpha = \star \alpha$$

for all q -form α .

2.4 Contact geometry

In this subsection some of basics of contact geometry is summarized. Roughly speaking, contact geometry is a geometry where a class of 1-forms are provided for manifolds. In this paper contact manifolds are used for describing implicit Hamiltonian systems.

Definition 2.19. (*Contact manifold*): Let \mathcal{C} be a $(2n + 1)$ -dimensional manifold, and λ a 1-form on \mathcal{C} such that

$$\lambda \wedge \underbrace{d\lambda \wedge d\lambda \cdots d\lambda}_n \neq 0,$$

at any point on \mathcal{C} . If \mathcal{C} carries λ , then (\mathcal{C}, λ) is referred to as a contact manifold and λ a contact form.

Remark 2.4. The $(2n + 1)$ -form $\lambda \wedge d\lambda \wedge \cdots \wedge d\lambda$ can be used for a volume form.

It should be noted that there are other definitions of contact manifold. However the definition above is used in this paper.

There is a standard local coordinate system.

Theorem 2.1. (*Existence of particular coordinates*): Given a contact manifold (\mathcal{C}, λ) , there exist local $(2n + 1)$ coordinates (x, y, z) with $x = \{x^1, \dots, x^n\}$ and $y = \{y_1, \dots, y_n\}$, in which λ has the form

$$\lambda = dz - y_a dx^a. \quad (1)$$

Definition 2.20. (*Canonical coordinates or Darboux coordinates*): The $(2n + 1)$ coordinates (x, y, z) introduced in Theorem 2.1 are referred to as the canonical coordinates, or the Darboux coordinates.

In addition to the above coordinates, ones in which λ has the form $\lambda = dz + y_a dx^a$ are also used in the literature. In this paper (1) is used.

Given a contact manifold, there exists a unique vector field that is defined as follows.

Definition 2.21. (*Reeb vector field or characteristic vector field*): Let (\mathcal{C}, λ) be a contact manifold, and \mathcal{R} a vector field on \mathcal{C} . If \mathcal{R} satisfies

$$\iota_{\mathcal{R}} d\lambda = 0, \quad \text{and} \quad \iota_{\mathcal{R}} \lambda = 1, \quad (2)$$

then $\mathcal{R} \in \Gamma TC$ is referred to as the Reeb vector field, or the characteristic vector field.

When λ is given, a coordinate expression for \mathcal{R} is given as follows.

Proposition 2.1. (*Coordinate expression of the Reeb vector field*): Let (\mathcal{C}, λ) be a contact manifold, and \mathcal{R} the Reeb vector field, (x, y, z) the canonical coordinates such that $\lambda = dz - y_a dx^a$ with $x = \{x^1, \dots, x^n\}$ and $y = \{y_1, \dots, y_n\}$. Then the coordinate expression of the Reeb vector field \mathcal{R} is given by

$$\mathcal{R} = \frac{\partial}{\partial z}. \quad (3)$$

The following submanifold plays various roles in applications of contact geometry.

Definition 2.22. (*Legendre submanifold*): Let (\mathcal{C}, λ) be a contact manifold, and \mathcal{A} a submanifold of \mathcal{C} . If \mathcal{A} is a maximal dimensional integral submanifold of λ , then \mathcal{A} is referred to as a Legendre submanifold.

The following theorem states the dimension of a Legendre submanifold for a given contact manifold.

Theorem 2.2. (*Maximal dimensional integral submanifold*): On a $(2n + 1)$ -dimensional contact manifold (\mathcal{C}, λ) , a maximal dimensional integral submanifold of λ is equal to n .

Combining Theorem 2.2 and Definition 2.22, one concludes the following.

Theorem 2.3. (*Dimension of Legendre submanifolds*): The dimension of any Legendre submanifold of a $(2n + 1)$ -dimensional contact manifold is n .

The following theorem shows the explicit local expressions of Legendre submanifolds in terms of canonical coordinates.

Theorem 2.4. (Local expressions of Legendre submanifolds, [1]): Let (C, λ) be a $(2n+1)$ -dimensional contact manifold, and (x, y, z) the canonical coordinates such that $\lambda = dz - y_a dx^a$ with $x = \{x^1, \dots, x^n\}$ and $y = \{y_1, \dots, y_n\}$. For any partition $I \cup J$ of the set of indices $\{1, \dots, n\}$ into two disjoint subsets I and J , and for a function $\phi(x^J, y_I)$ of n variables $y_i, i \in I$, and $x^j, j \in J$ the $(n+1)$ equations

$$x^i = -\frac{\partial \phi}{\partial y_i}, \quad y_j = \frac{\partial \phi}{\partial x^j}, \quad z = \phi - y_i \frac{\partial \phi}{\partial y_i}, \quad (4)$$

define a Legendre submanifold. Conversely, every Legendre submanifold of (C, λ) in a neighborhood of any point is defined by these equations for at least one of the 2^n possible choices of the subset I .

The function ϕ in (4) is often used in this paper, and then that has a special name as follows.

Definition 2.23. (Legendre submanifold generated by a function): The function ϕ used in Theorem 2.4 is referred to as a generating function of the Legendre submanifold. If a Legendre submanifold \mathcal{A} is expressed as (4), then \mathcal{A} is referred to as a Legendre submanifold generated by ϕ .

The following are examples of local expressions for Legendre submanifolds.

Example 2.1. Let (C, λ) be a $(2n+1)$ -dimensional contact manifold, (x, y, z) the canonical coordinates such that $\lambda = dz - y_a dx^a$ with $x = \{x^1, \dots, x^n\}$ and $y = \{y_1, \dots, y_n\}$, and ψ a function of x only. The Legendre submanifold \mathcal{A}_ψ generated by ψ with $\Phi_{C\mathcal{A}_\psi} : \mathcal{A}_\psi \rightarrow C$ being the embedding is such that

$$\Phi_{C\mathcal{A}_\psi} \mathcal{A}_\psi = \left\{ (x, y, z) \in C \mid y_j = \frac{\partial \psi}{\partial x^j}, \text{ and } z = \psi(x), \quad j \in \{1, \dots, n\} \right\}. \quad (5)$$

One can verify that $\Phi_{C\mathcal{A}_\psi}^* \lambda = 0$.

Example 2.2. Let (C, λ) be a $(2n+1)$ -dimensional contact manifold, (x, y, z) the canonical coordinates such that $\lambda = dz - y_a dx^a$ with $x = \{x^1, \dots, x^n\}$ and $y = \{y_1, \dots, y_n\}$, and φ a function of y only. The Legendre submanifold \mathcal{A}_φ generated by $-\varphi$ with $\Phi_{C\mathcal{A}_\varphi} : \mathcal{A}_\varphi \rightarrow C$ being the embedding is such that

$$\Phi_{C\mathcal{A}_\varphi} \mathcal{A}_\varphi = \left\{ (x, y, z) \in C \mid x^i = \frac{\partial \varphi}{\partial y_i}, \text{ and } z = y_i \frac{\partial \varphi}{\partial y_i} - \varphi(y), \quad i \in \{1, \dots, n\} \right\}. \quad (6)$$

One can verify that $\Phi_{C\mathcal{A}_\varphi}^* \lambda = 0$.

One can choose a function ψ in Example 2.1 to generate \mathcal{A}_ψ , and choose a function φ in Example 2.2 to generate \mathcal{A}_φ independently, and in this case there is no relation between \mathcal{A}_ψ and \mathcal{A}_φ in general. On the other hand, when ψ is strictly convex, and φ is carefully chosen, it can be shown that there is a relation between \mathcal{A}_ψ and \mathcal{A}_φ . To discuss such a case, the following transform should be introduced. The convention is adapted to that in information geometry. Note that several conventions exist in the literature.

Definition 2.24. (Total Legendre transform): Let \mathcal{M} be an n -dimensional manifold, $x = \{x^1, \dots, x^n\}$ coordinates, and ψ a function of x . Then the total Legendre transform of ψ with respect to x is defined to be

$$\mathfrak{L}[\psi](y) := \sup_x [x^a y_a - \psi(x)], \quad (7)$$

where $y = \{y_1, \dots, y_n\}$.

From this definition, one has several formulae that will be used in the following sections. To state these formulae, one defines strictly convex function and convexity as follows.

Definition 2.25. (Strictly convex function and convexity): Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a convex domain, and f a function of $\{x^a\}$ on \mathcal{A}_0 . If the matrix

$$\frac{\partial^2 f}{\partial x^a \partial x^b},$$

is strictly positive definite, then the function f is referred to as a strictly convex function. In addition, the property that f is strictly convex is referred to as convexity.

Then one has the following.

Theorem 2.5. (Formulae involving the total Legendre transform): Let \mathcal{M} be an n -dimensional manifold, $x = \{x^1, \dots, x^n\}$ coordinates, $\psi \in \Gamma\Lambda^0\mathcal{M}$ a strictly convex function of x only, and φ the function of y obtained by the total Legendre transform of ψ with respect to x where $y = \{y_1, \dots, y_n\} : \varphi(y) = \mathfrak{L}[\psi](y)$. Then, for each a and fixed y , the equation

$$y_a = \left. \frac{\partial\psi(x)}{\partial x^a} \right|_{x=x_*} = \frac{\partial\psi(x_*)}{\partial x_*^a},$$

has the unique solution $x_*^a = x_*^a(y)$, ($a \in \{1, \dots, n\}$). In addition it follows that

$$\varphi(y) = x_*^a y_a - \psi(x_*), \quad \frac{\partial\varphi}{\partial y_a} = x_*^a, \quad \delta_b^a = \frac{\partial^2\psi}{\partial x_*^b \partial x_*^a} \frac{\partial^2\varphi}{\partial y_a \partial y_b},$$

and

$$\det\left(\frac{\partial^2\psi}{\partial x^a \partial x^b}\right) > 0, \quad \det\left(\frac{\partial^2\varphi}{\partial y_a \partial y_b}\right) > 0.$$

A way to describe dynamics on a contact manifold is to introduce a continuous diffeomorphism with a parameter. First, one defines a class of diffeomorphisms on contact manifolds.

Definition 2.26. (Contact diffeomorphism): Let (\mathcal{C}, λ) be a $(2n+1)$ -dimensional contact manifold, and $\Phi : \mathcal{C} \rightarrow \mathcal{C}$ a diffeomorphism. If it follows that

$$\Phi^*\lambda = f\lambda,$$

where $f \in \Gamma\Lambda^0\mathcal{C}$ is a function that does not vanish at any point of \mathcal{C} , then the map Φ is referred to as a contact diffeomorphism.

Remark 2.5. It follows that Φ preserves the contact structure, $\ker \lambda := \{X \in \Gamma TC \mid \iota_X \lambda = 0\}$.

In addition to this diffeomorphism, one can introduce one-parameter groups as follows.

Definition 2.27. (One-parameter group of continuous contact transformations): Let (\mathcal{C}, λ) be a $(2n+1)$ -dimensional contact manifold, and $\Phi_t : \mathcal{C} \rightarrow \mathcal{C}$ a diffeomorphism with $t \in \mathbb{R}$ that satisfies $\Phi_0 = \text{Id}_{\mathcal{C}}$ and $\Phi_{t+s} = \Phi_t \circ \Phi_s$, ($t, s \in \mathbb{R}$), where $\text{Id}_{\mathcal{C}}$ is such that $\text{Id}_{\mathcal{C}}\xi = \xi$ for all $\xi \in \mathcal{C}$. If it follows that

$$\Phi_t^*\lambda = f_t\lambda,$$

where $f_t \in \Gamma\Lambda^0\mathcal{C}$ is a function that does not vanish at any point of \mathcal{C} , then the Φ_t is referred to as a one-parameter group of continuous contact transformations. If $t, s \in \mathbb{T}$ with some $\mathbb{T} \subset \mathbb{R}$, then Φ_t is referred to as a one-parameter local transformation group of continuous contact transformations.

A contact vector field is defined as follows.

Definition 2.28. (Contact vector field): Let (\mathcal{C}, λ) be a contact manifold, and X a vector field on \mathcal{C} . If X satisfies

$$\mathcal{L}_X \lambda = f\lambda,$$

where f is a function on \mathcal{C} , then X is referred to as a contact vector field.

A one-parameter (local) transformation groups is realized by integrating the following vector field.

Definition 2.29. (Contact vector field associated to a contact Hamiltonian): Let (\mathcal{C}, λ) be a contact manifold, \mathcal{R} the Reeb vector field, h a function on \mathcal{C} , and X_h a vector field. If $X_h \in \Gamma TC$ satisfies

$$\iota_{X_h} \lambda = h, \quad \text{and} \quad \iota_{X_h} d\lambda = -(\text{d}h - (\mathcal{R}h)\lambda), \quad (8)$$

then X_h is referred to as a contact vector field associated to a function h or a contact Hamiltonian vector field. In addition h is referred to as a contact Hamiltonian.

Remark 2.6. The definition (8) and the Cartan formula give

$$\mathcal{L}_{X_h} \lambda = (\mathcal{R}h) \lambda. \quad (9)$$

Remark 2.7. With (8), (9), and the formula $\mathcal{L}_X \iota_X \alpha = \iota_X \mathcal{L}_X \alpha$ for an arbitrary vector field X and an arbitrary q -form α with $q \in \{0, 1, \dots\}$, one has that

$$X_h h = \mathcal{L}_{X_h} h = \mathcal{L}_{X_h} (\iota_{X_h} \lambda) = \iota_{X_h} (\mathcal{L}_{X_h} \lambda) = (\mathcal{R}h) (\iota_{X_h} \lambda) = (\mathcal{R}h) h. \quad (10)$$

Thus, unlike the case of autonomous Hamiltonian systems, h is not conserved.

From (9) and Definition 2.28, one has the following.

Theorem 2.6. (*Relation between a contact Hamiltonian vector field and a contact vector field*): Let (\mathcal{C}, λ) be a contact manifold, h a contact Hamiltonian, and X_h a contact Hamiltonian vector field. Then X_h is a contact vector field.

Local expressions of a contact Hamiltonian vector field (8) are calculated as follows.

Proposition 2.2. (*Local expression of contact Hamiltonian vector field*): Let (\mathcal{C}, λ) be a $(2n+1)$ -dimensional contact manifold, h a contact Hamiltonian, X_h a contact Hamiltonian vector field, and (x, y, z) the canonical coordinates such that $\lambda = dz - y_a dx^a$ with $x = \{x^1, \dots, x^n\}$ and $y = \{y_1, \dots, y_n\}$. Then

$$X_h = \dot{x}^a \frac{\partial}{\partial x^a} + \dot{y}_a \frac{\partial}{\partial y_a} + \dot{z} \frac{\partial}{\partial z},$$

where $\dot{\cdot}$ denotes the differential with respect to a parameter $t \in \mathbb{R}$, or $t \in \mathbb{T}$ with some $\mathbb{T} \subset \mathbb{R}$, and

$$\dot{x}^a = -\frac{\partial h}{\partial y_a}, \quad \dot{y}_a = \frac{\partial h}{\partial x^a} + y_a \frac{\partial h}{\partial z}, \quad \dot{z} = h - y_a \frac{\partial h}{\partial y_a}, \quad a \in \{1, \dots, n\}. \quad (11)$$

Remark 2.8. When $\lambda = dz + y_a dx^a$, signs in (11) are changed.

Let ϕ_h be an integral curve of X_h such that $\phi_h : \mathbb{T} \rightarrow \mathcal{C}, (t \mapsto (x, y, z))$ with some $\mathbb{T} \subseteq \mathbb{R}$. Then $\dot{\cdot}$ denotes the derivative with respect to $t \in \mathbb{T}$. Physically this t is interpreted as time. In this case contact vector fields can be viewed as dynamical systems. Throughout this paper, vector fields are always identified with dynamical systems, and integral curves are focused when vector fields are given.

The following theorem is well-known, and has been used in the literature of geometric thermodynamics.

Theorem 2.7. (*Tangent vector field of a Legendre submanifold realized by a contact Hamiltonian vector field, [32]*): Let (\mathcal{C}, λ) be a contact manifold, \mathcal{A} a Legendre submanifold, and h a contact Hamiltonian. Then the contact Hamiltonian vector field is tangent to \mathcal{A} if and only if h vanishes on \mathcal{A} .

Introducing some symbols, one has other equivalent expressions for the Legendre submanifolds (5) and (6). The following functions were introduced in Ref. [16], and it was shown that the introduced functions are tools to describe contact Hamiltonian vector fields concisely. They are as follows.

Definition 2.30. (*Adapted functions, [16]*): Let (\mathcal{C}, λ) be a $(2n+1)$ -dimensional contact manifold, and (x, y, z) canonical coordinates such that $\lambda = dz - y_a dx^a$ with $x = \{x^1, \dots, x^n\}$ and $y = \{y_1, \dots, y_n\}$. In addition let ψ be a function on \mathcal{C} depending on x only, and φ a function on \mathcal{C} depending on y only. Then the functions $\Delta_0^\psi, \{\Delta_1^\psi, \dots, \Delta_n^\psi\} : \mathcal{C} \rightarrow \mathbb{R}$ and $\Delta_\varphi^0, \{\Delta_\varphi^1, \dots, \Delta_\varphi^n\} : \mathcal{C} \rightarrow \mathbb{R}$ such that

$$\Delta_0^\psi(x, z) := \psi(x) - z, \quad \Delta_a^\psi(x, y) := \frac{\partial \psi}{\partial x^a} - y_a, \quad a \in \{1, \dots, n\}.$$

$$\Delta_\varphi^0(x, y, z) := x^j y_j - \varphi(y) - z, \quad \Delta_\varphi^a(x, y) := x^a - \frac{\partial \varphi}{\partial y_a}, \quad a \in \{1, \dots, n\}.$$

are referred to as adapted functions.

In adapted functions, the local expressions of Legendre submanifolds generated by ψ and those by $-\varphi$ can be written as follows[16].

Proposition 2.3. (Local expressions of Legendre submanifold with adapted functions, [16]): The Legendre submanifold \mathcal{A}_ψ generated by ψ as in (5) is expressed as

$$\Phi_{\mathcal{CA}\psi}\mathcal{A}_\psi = \left\{ (x, y, z) \in \mathcal{C} \mid \Delta_0^\psi = 0 \text{ and } \Delta_1^\psi = \cdots = \Delta_n^\psi = 0 \right\},$$

where $\Phi_{\mathcal{CA}\psi}\mathcal{A}_\psi : \mathcal{A}_\psi \rightarrow \mathcal{C}$ is the embedding. Similarly, the Legendre submanifold \mathcal{A}_φ generated by $-\varphi$ as in (6) is expressed as

$$\Phi_{\mathcal{CA}\varphi}\mathcal{A}_\varphi = \left\{ (x, y, z) \in \mathcal{C} \mid \Delta_\varphi^0 = 0 \text{ and } \Delta_\varphi^1 = \cdots = \Delta_\varphi^n = 0 \right\},$$

where $\Phi_{\mathcal{CA}\varphi}\mathcal{A}_\varphi : \mathcal{A}_\varphi \rightarrow \mathcal{C}$ is the embedding.

From this proposition, a Legendre submanifold $\Phi_{\mathcal{CA}\psi}\mathcal{A}_\psi$ is a submanifold where the constraints $\Delta_0^\psi = \cdots = \Delta_n^\psi$ hold. Vector fields on $\Phi_{\mathcal{CA}\psi}\mathcal{A}_\psi$ can be constructed with a restricted contact Hamiltonian vector field. It was shown in Ref.[16] that contact Hamiltonian vector fields are also written in terms of adapted functions.

Proposition 2.4. (Restricted contact Hamiltonian vector field as the push-forward of a vector field on the Legendre submanifold generated by ψ , [16]): Let $\{F_\psi^1, \dots, F_\psi^n\}$ be a set of functions of x on \mathcal{A}_ψ such that they do not identically vanish, and $\tilde{X}_\psi^0 \in T_x\mathcal{A}_\psi, (x \in \mathcal{A}_\psi)$ the vector field given as

$$\tilde{X}_\psi^0 = \dot{x}^a \frac{\partial}{\partial x^a}, \quad \text{where } \dot{x}^a = F_\psi^a(x), \quad (a \in \{1, \dots, n\}).$$

In addition, let $X_\psi^0 := (\Phi_{\mathcal{CA}\psi})_* \tilde{X}_\psi^0 \in T_\xi \mathcal{A}_\psi^C, (\xi \in \mathcal{A}_\psi^C)$ be the push-forward of \tilde{X}_ψ^0 , where $\mathcal{A}_\psi^C := \Phi_{\mathcal{CA}\psi}\mathcal{A}_\psi$ with $\Phi_{\mathcal{CA}\psi} : \mathcal{A}_\psi \rightarrow \mathcal{C}$ being the embedding :

$$\begin{aligned} \Phi_{\mathcal{CA}\psi} &: \mathcal{A}_\psi \rightarrow \mathcal{A}_\psi^C, & x &\mapsto (x, y(x), z(x)) \\ (\Phi_{\mathcal{CA}\psi})_* &: T_x \mathcal{A}_\psi \rightarrow T_\xi \mathcal{A}_\psi^C, & \tilde{X}_\psi^0 &\mapsto X_\psi^0. \end{aligned}$$

Then it follows that

$$X_\psi^0 = \dot{x}^a \frac{\partial}{\partial x^a} + \dot{y}_a \frac{\partial}{\partial y_a} + \dot{z} \frac{\partial}{\partial z}, \quad \text{where } \dot{x}^a = F_\psi^a(x), \quad \dot{y}_a = \frac{d}{dt} \left(\frac{\partial \psi}{\partial x^a} \right), \quad \dot{z} = \frac{d\psi}{dt}. \quad (12)$$

In addition, one has that $X_\psi^0 = X_{h_\psi}|_{h_\psi=0}$. Here X_{h_ψ} is the contact Hamiltonian vector field associated with

$$h_\psi(x, y, z) = \Delta_a(x, y) F_\psi^a(x) + \Gamma_\psi(\Delta_0(x, z)), \quad (13)$$

where Γ_ψ is a function of Δ_0 such that

$$\Gamma_\psi(\Delta_0) = \begin{cases} 0 & \text{for } \Delta_0 = 0 \\ \text{non-zero} & \text{for } \Delta_0 \neq 0 \end{cases}.$$

Remark 2.9. The functions $\{F_\psi^1, \dots, F_\psi^n\}$ need not depend on ψ .

Remark 2.10. The value of the generating function ψ is not conserved along this restricted contact Hamiltonian vector field, since

$$\mathcal{L}_{X_\psi^0} \psi = \frac{\partial \psi}{\partial x^a} \frac{dx^a}{dt} = \frac{\partial \psi}{\partial x^a} F_\psi^a,$$

does not identically vanish in general. In some cases, this vanishes. Consider the system with $n = 2$, and $F_\psi^1 = \partial \psi / \partial x^2, F_\psi^2 = -\partial \psi / \partial x^1$. Then $\mathcal{L}_{X_\psi^0} \psi = 0$.

There exists a counterpart of Proposition 2.4 as follows.

Proposition 2.5. (Restricted contact Hamiltonian vector field as the push-forward of vector fields on the Legendre submanifold generated by $-\varphi$, [16]): Let $\{F_1^\varphi, \dots, F_n^\varphi\}$ be a set of functions of y on \mathcal{A}_φ such that they do not identically vanish, and $\tilde{X}_\varphi^0 \in T_y \mathcal{A}_\varphi, (y \in \mathcal{A}_\varphi)$ given as

$$\tilde{X}_\varphi^0 = \dot{y}_a \frac{\partial}{\partial y_a}, \quad \text{where } \dot{y}_a = F_a^\varphi(y).$$

In addition, let $X_\varphi^0 := (\Phi_{\mathcal{CA}_\varphi})_* \tilde{X}_\varphi^0 \in T_\xi \mathcal{A}_\varphi^C, (\xi \in \mathcal{A}_\varphi^C)$ be the push-forward of \tilde{X}_φ^0 , where $\mathcal{A}_\varphi^C := \Phi_{\mathcal{CA}_\varphi} \mathcal{A}_\varphi$ with $\Phi_{\mathcal{CA}_\varphi} : \mathcal{A}_\varphi \rightarrow \mathcal{C}$ being the embedding :

$$\begin{aligned} \Phi_{\mathcal{CA}_\varphi} &: \mathcal{A}_\varphi \rightarrow \mathcal{A}_\varphi^C, & y &\mapsto (x(y), y, z(y)) \\ (\Phi_{\mathcal{CA}_\varphi})_* &: T_y \mathcal{A}_\varphi \rightarrow T_\xi \mathcal{A}_\varphi^C, & \tilde{X}_\varphi^0 &\mapsto X_\varphi^0. \end{aligned}$$

Then it follows that

$$X_\varphi^0 = \dot{x}^a \frac{\partial}{\partial x^a} + \dot{y}_a \frac{\partial}{\partial y_a} + \dot{z} \frac{\partial}{\partial z}, \quad \text{where} \quad \dot{x}_a = \frac{d}{dt} \left(\frac{\partial \varphi}{\partial y_a} \right), \quad \dot{y}^a = F_a^\varphi(y), \quad \dot{z} = y_j F_k^\varphi \frac{\partial^2 \varphi}{\partial y_k \partial y_j}. \quad (14)$$

In addition, one has that $X_\varphi^0 = X_{h_\varphi}|_{h_\varphi=0}$. Here X_{h_φ} is the contact Hamiltonian vector field associated with

$$h_\varphi(x, y) = \Delta^a(x, y) F_a^\varphi(y) + \Gamma^\varphi(\Delta^0), \quad (15)$$

where Γ^φ is a function of Δ^0 such that

$$\Gamma^\varphi(\Delta^0) = \begin{cases} 0 & \text{for } \Delta^0 = 0 \\ \text{non-zero} & \text{for } \Delta^0 \neq 0 \end{cases}.$$

Remark 2.11. The functions $\{F_1^\varphi, \dots, F_n^\varphi\}$ need not depend on φ .

Remark 2.12. The value of the generating function φ is not conserved along this restricted contact Hamiltonian vector field, since

$$\mathcal{L}_{X_\varphi^0} \varphi = \frac{\partial \varphi}{\partial y_a} \frac{dy_a}{dt} = \frac{\partial \varphi}{\partial y_a} F_a^\varphi,$$

does not identically vanish in general. In some cases, this vanishes. Consider the system with $n = 2$, and $F_1^\varphi = \partial \varphi / \partial y_2, F_2^\varphi = -\partial \varphi / \partial y_1$. Then $\mathcal{L}_{X_\varphi^0} \varphi = 0$.

2.5 Contact manifold over base space

In this paper it is shown that distributed port-Hamiltonian systems are written in terms of a contact manifold over some base space, and the contact manifold over a base space is treated as a bundle. To this end, some basic definitions and theorems obtained from the standard contact geometry are summarized below. The discussions in this subsection are based on Ref. [17].

Definition 2.31. (Contact manifold over base space or contact bundle): Let \mathcal{B} be a $d_{\mathcal{B}}$ -dimensional manifold, $(\mathcal{K}, \pi, \mathcal{B})$ a bundle over the base space \mathcal{B} , the fiber space $\pi^{-1}(\zeta)$ at a point ζ of \mathcal{B} a $(2n+1)$ -dimensional manifold \mathcal{C}_ζ , $\mathcal{K} = \bigcup_{\zeta \in \mathcal{B}} \mathcal{C}_\zeta$, and the structure group G a contact transformation group. If \mathcal{K} carries a vertical form $\lambda_{\mathbb{V}}$ such that

$$\lambda_{\mathbb{V}} \wedge \underbrace{d_{\mathbb{V}} \lambda_{\mathbb{V}} \wedge \dots \wedge d_{\mathbb{V}} \lambda_{\mathbb{V}}}_n \neq 0, \quad \text{at each point of } \pi^{-1}(\zeta) \text{ at each point } \zeta \text{ of } \mathcal{B}$$

then \mathcal{C}_ζ is referred to as a $((2n+1)$ -dimensional) contact manifold on the fiber space $\pi^{-1}(\zeta)$, $(\zeta \in \mathcal{B})$, the quadruplet $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ is referred to as a $((2n+1)$ -dimensional) contact manifold over the base space \mathcal{B} or a contact bundle, and $\lambda_{\mathbb{V}}$ a contact vertical form.

In this paper trivial bundles are only considered, then the transition functions are always identical since this simple case is enough for our contact geometric formulation of distributed-parameter port-Hamiltonian systems. The contact geometry of the vertical space is the same as the standard contact geometry. Thus, all of the definitions and theorems for the standard contact geometry can be brought to vertical spaces. They are shown below.

At each base point ζ of \mathcal{B} , one has Darboux's theorem for $\pi^{-1}(\zeta)$. Therefore one has the following.

Theorem 2.8. (Existence of Darboux coordinates on fiber space): For the $(2n+1)$ -dimensional contact manifold over a base space $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$, there exist local coordinates (x, y, z) for $\pi^{-1}(\zeta)$ with $x = \{x^1, \dots, x^n\}$ and $y = \{y_1, \dots, y_n\}$ in which $\lambda_{\mathbb{M}}^q \in \Gamma \Lambda_{\mathbb{H}, \mathbb{V}}^{q,1} \mathcal{K}$ has the form

$$\lambda_{\mathbb{M}}^q = \rho^q \wedge \lambda_{\mathbb{V}}, \quad \text{where} \quad \lambda_{\mathbb{V}} = dz - y_a dx^a,$$

with some $\rho^q \in \Gamma \Lambda_{\mathbb{H}}^q \mathcal{K}$ being nowhere vanishing.

Definition 2.32. (Canonical coordinates or Darboux coordinates): The $(2n + 1)$ coordinates introduced in Theorem 2.8 are referred to as the canonical coordinates for a fiber space or the Darboux coordinates for a fiber space.

Definition 2.33. (Canonical contact mixed form and canonical contact vertical form): Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a contact manifold over a base space \mathcal{B} , and ρ^q a nowhere vanishing horizontal q -form. A mixed $(q, 1)$ -form $\lambda_{\mathbb{M}}^q \in \Gamma \Lambda_{\mathbb{H}, \mathbb{V}}^{q, 1} \mathcal{K}$ written as

$$\lambda_{\mathbb{M}}^q = \rho^q \wedge \lambda_{\mathbb{V}}, \quad \text{where} \quad \lambda_{\mathbb{V}} = d_{\mathbb{V}} z - y_a d_{\mathbb{V}} x^a,$$

is referred to as the canonical contact mixed $(q, 1)$ -form associated with ρ^q , and $\lambda_{\mathbb{V}} \in \Gamma \Lambda_{\mathbb{V}}^1 \mathcal{K}$ the canonical contact vertical form.

Definition 2.34. (Reeb vertical vector field): Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a contact manifold over a base space \mathcal{B} , and $\mathcal{R}_{\mathbb{V}}$ a vertical vector field on \mathcal{K} . If $\mathcal{R}_{\mathbb{V}}$ satisfies

$$\iota_{\mathcal{R}_{\mathbb{V}}} \lambda_{\mathbb{V}} = 1, \quad \text{and} \quad \iota_{\mathcal{R}_{\mathbb{V}}} d_{\mathbb{V}} \lambda_{\mathbb{V}} = 0,$$

then $\mathcal{R}_{\mathbb{V}}$ is referred to as the Reeb vertical vector field on \mathcal{K} .

Proposition 2.6. (Coordinate expression of the Reeb vertical vector field): Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a contact manifold over a base space \mathcal{B} , $\mathcal{R}_{\mathbb{V}}$ the Reeb vertical vector field on \mathcal{K} , and (x, y, z) the canonical coordinates for the fiber space such that $\lambda_{\mathbb{V}} = d_{\mathbb{V}} z - y_a d_{\mathbb{V}} x^a$. Then the coordinate expression of the Reeb vertical vector field $\mathcal{R}_{\mathbb{V}}$ is

$$\mathcal{R}_{\mathbb{V}} = \frac{\partial}{\partial z}.$$

Definition 2.35. (Contact Hamiltonian vertical vector field): Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a contact manifold over a base space \mathcal{B} with $\dim \mathcal{B} = d_{\mathcal{B}}$, $\mathcal{B}_0 \subseteq \mathcal{B}$ a $d_{\mathcal{B}}$ -dimensional space, $\rho^{d_{\mathcal{B}}} \in \Gamma \Lambda_{\mathbb{H}}^{d_{\mathcal{B}}} \mathcal{K}$ a nowhere vanishing horizontal form, $\mathcal{R}_{\mathbb{V}}$ the Reeb vertical vector field on \mathcal{K} , $\tilde{h} \in \Gamma F \mathcal{K}$ the functional given by

$$\tilde{h} = \int_{\mathcal{B}_0} h \rho^{d_{\mathcal{B}}},$$

with some $h \in \Gamma \Lambda_{\mathbb{V}}^0 \mathcal{K}$, and $X_{\tilde{h}}$ a vertical vector field on \mathcal{K} . If $X_{\tilde{h}}$ satisfies

$$\iota_{X_{\tilde{h}}} \lambda_{\mathbb{V}} = h \quad \text{and} \quad \iota_{X_{\tilde{h}}} d_{\mathbb{V}} \lambda_{\mathbb{V}} = - (d_{\mathbb{V}} h - (\mathcal{R}_{\mathbb{V}} h) \lambda_{\mathbb{V}}), \quad (16)$$

then $X_{\tilde{h}}$ is referred to as the contact Hamiltonian vertical vector field, \tilde{h} a contact Hamiltonian functional, and h a contact Hamiltonian vertical function.

Remark 2.13. With the Cartan formula, one has that $\mathcal{L}_{X_{\tilde{h}}} \lambda_{\mathbb{V}} = (\mathcal{R}_{\mathbb{V}} h) \lambda_{\mathbb{V}}$. Thus, $\mathcal{L}_{X_{\tilde{h}}} \lambda_{\mathbb{M}}^{d_{\mathcal{B}}} = \rho^{d_{\mathcal{B}}} \wedge \mathcal{L}_{X_{\tilde{h}}} \lambda_{\mathbb{V}} = (\mathcal{R}_{\mathbb{V}} h) \lambda_{\mathbb{M}}^{d_{\mathcal{B}}}$.

In the following the coordinate expression of a contact Hamiltonian vertical vector field is shown.

Proposition 2.7. (Coordinate expression of a contact vertical Hamiltonian vector field): Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a contact manifold over a base space \mathcal{B} with $\dim \mathcal{B} = d_{\mathcal{B}}$, $\mathcal{B}_0 \subseteq \mathcal{B}$ a $d_{\mathcal{B}}$ -dimensional space, $\rho^{d_{\mathcal{B}}} \in \Gamma \Lambda_{\mathbb{H}}^{d_{\mathcal{B}}} \mathcal{K}$ a nowhere vanishing form, (x, y, z) the canonical coordinates for the fiber space such that $\lambda_{\mathbb{V}} = d_{\mathbb{V}} z - y_a d_{\mathbb{V}} x^a$ with $x = \{x^1, \dots, x^n\}$ and $y = \{y_1, \dots, y_n\}$, \tilde{h} the contact Hamiltonian functional given by

$$\tilde{h} = \int_{\mathcal{B}_0} h \rho^{d_{\mathcal{B}}},$$

with some $h \in \Gamma \Lambda_{\mathbb{V}}^0 \mathcal{K}$ depending on (x, y, z) , and $X_{\tilde{h}}$ the contact Hamiltonian vertical vector field on \mathcal{K} . Then, the canonical coordinate expression of (16) is given as

$$X_{\tilde{h}} = \dot{x}^a \frac{\partial}{\partial x^a} + \dot{y}_a \frac{\partial}{\partial y_a} + \dot{z} \frac{\partial}{\partial z},$$

where

$$\dot{x}^a = -\frac{\partial h}{\partial y_a}, \quad \dot{y}_a = \frac{\partial h}{\partial x^a} + y_a \frac{\partial h}{\partial z}, \quad \dot{z} = h - y_a \frac{\partial h}{\partial y_a}, \quad (17)$$

or equivalently,

$$\dot{x}^a = -\frac{\delta \tilde{h}}{\delta y_a}, \quad \dot{y}_a = \frac{\delta \tilde{h}}{\delta x^a} + y_a \frac{\delta \tilde{h}}{\delta z}, \quad \dot{z} = h - y_a \frac{\delta \tilde{h}}{\delta y_a}.$$

Remark 2.14. The coordinate expression (17) is formally the same as that of (11).

Analogous to Definition 2.22, Legendre submanifold on a bundle is defined as follows.

Definition 2.36. (*Legendre submanifold of vertical space and that of fiber space*) : Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a contact manifold over a base space \mathcal{B} . If \mathcal{A}_{ζ} is a maximal dimensional integral submanifold of $\lambda_{\mathbb{V}}$ on $\pi^{-1}(\zeta)$, $(\zeta \in \mathcal{B})$, then \mathcal{A}_{ζ} is referred to as a Legendre submanifold in the fiber space $\pi^{-1}(\zeta)$, and $\mathcal{A}^{\mathcal{K}} = \bigcup_{\zeta \in \mathcal{B}} \mathcal{A}_{\zeta}$ a Legendre submanifold in the fiber space.

An analogous theorem from Theorem 2.4 holds for the present bundles. Examples of Legendre submanifolds on fiber spaces are as follows.

Example 2.3. Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a $(2n+1)$ -dimensional contact manifold over a base space \mathcal{B} with $\dim \mathcal{B} = d_{\mathcal{B}}$, (x, y, z) the canonical coordinates for the fiber space such that $\lambda_{\mathbb{V}} = d_{\mathbb{V}}z - y_a d_{\mathbb{V}}x^a$ with $x = \{x^1, \dots, x^n\}$ and $y = \{y_1, \dots, y_n\}$, $\psi \in \Gamma \Lambda_{\mathbb{V}}^0 \mathcal{K}$ a vertical function of x , $\rho^{d_{\mathcal{B}}}$ a nowhere vanishing horizontal $d_{\mathcal{B}}$ -form, $\mathcal{B}_0 \subseteq \mathcal{B}$ a $d_{\mathcal{B}}$ -dimensional space, and $\tilde{\psi}_{\mathcal{B}_0} \in \Gamma F\mathcal{K}$ the functional

$$\tilde{\psi}_{\mathcal{B}_0} = \int_{\mathcal{B}_0} \psi \rho^{d_{\mathcal{B}}}.$$

Then, the Legendre submanifold $\mathcal{A}_{\zeta\psi}$ generated by ψ on $\pi^{-1}(\zeta)$, $(\zeta \in \mathcal{B})$ with $\Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\psi}} : \mathcal{A}_{\zeta\psi} \rightarrow \mathcal{C}_{\zeta}$ being the embedding is such that

$$\Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\psi}} \mathcal{A}_{\zeta\psi} = \left\{ (x, y, z) \in \mathcal{C}_{\zeta} \mid y_j = \frac{\partial \psi}{\partial x^j}, \text{ and } z = \psi(x), \quad j \in \{1, \dots, n\} \right\}. \quad (18)$$

This can also be written as

$$\Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\psi}} \mathcal{A}_{\zeta\psi} = \left\{ (x, y, z) \in \mathcal{C}_{\zeta} \mid y_j = \frac{\delta \tilde{\psi}_{\mathcal{B}_0}}{\delta x^j}, \text{ and } z = \psi(x), \quad j \in \{1, \dots, n\} \right\}.$$

In addition, $(\mathcal{A}_{\psi}^{\mathcal{K}}, \pi|_{\mathcal{A}_{\psi}^{\mathcal{K}}}, \mathcal{B})$ is a sub-bundle of $(\mathcal{K}, \pi, \mathcal{B})$, where

$$\mathcal{A}_{\psi}^{\mathcal{K}} = \bigcup_{\zeta \in \mathcal{B}} \Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\psi}} \mathcal{A}_{\zeta\psi}.$$

Example 2.4. Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a $(2n+1)$ -dimensional contact manifold over a base space \mathcal{B} , (x, y, z) the canonical coordinates for the fiber space such that $\lambda_{\mathbb{V}} = d_{\mathbb{V}}z - y_a d_{\mathbb{V}}x^a$ with $x = \{x^1, \dots, x^n\}$ and $y = \{y_1, \dots, y_n\}$, $\varphi \in \Gamma \Lambda_{\mathbb{V}}^0 \mathcal{K}$ a vertical function of y , $\rho^{d_{\mathcal{B}}}$ a nowhere vanishing horizontal $d_{\mathcal{B}}$ -form, $\mathcal{B}_0 \subseteq \mathcal{B}$ a $d_{\mathcal{B}}$ -dimensional space, and $\tilde{\varphi}_{\mathcal{B}_0} \in \Gamma F\mathcal{K}$ the functional

$$\tilde{\varphi}_{\mathcal{B}_0} = \int_{\mathcal{B}_0} \varphi \rho^{d_{\mathcal{B}}}.$$

Then, the Legendre submanifold $\mathcal{A}_{\zeta\varphi}$ generated by $-\varphi$ on $\pi^{-1}(\zeta)$, $(\zeta \in \mathcal{B})$ with $\Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\varphi}} : \mathcal{A}_{\zeta\varphi} \rightarrow \mathcal{C}_{\zeta}$ being the embedding is such that

$$\Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\varphi}} \mathcal{A}_{\zeta\varphi} = \left\{ (x, y, z) \in \mathcal{C}_{\zeta} \mid x_i = \frac{\partial \varphi}{\partial y_i}, \text{ and } z = y_i \frac{\partial \varphi}{\partial y_i} - \varphi(y), \quad i \in \{1, \dots, n\} \right\}. \quad (19)$$

This can also be written as

$$\Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\varphi}} \mathcal{A}_{\zeta\varphi} = \left\{ (x, y, z) \in \mathcal{C}_{\zeta} \mid x_i = \frac{\delta \tilde{\varphi}_{\mathcal{B}_0}}{\delta y_i}, \text{ and } z = y_i \frac{\delta \tilde{\varphi}_{\mathcal{B}_0}}{\delta y_i} - \varphi(y), \quad i \in \{1, \dots, n\} \right\}.$$

In addition, $(\mathcal{A}_\varphi^\mathcal{K}, \pi|_{\mathcal{A}_\varphi^\mathcal{K}}, \mathcal{B})$ is a sub-bundle of $(\mathcal{K}, \pi, \mathcal{B})$, where

$$\mathcal{A}_\varphi^\mathcal{K} = \bigcup_{\zeta \in \mathcal{B}} \Phi_{\mathcal{C}_\zeta \mathcal{A}_{\zeta\varphi}} \mathcal{A}_{\zeta\varphi}.$$

The total Legendre transform of a functional is defined as follows in this paper.

Definition 2.37. (*Total Legendre transform of functional*): Let $(\mathcal{K}, \lambda_\mathbb{V}, \pi, \mathcal{B})$ be a contact manifold over a base space \mathcal{B} with $\dim \mathcal{B} = d_\mathcal{B}$, $\mathcal{B}_0 \subseteq \mathcal{B}$ a $d_\mathcal{B}$ -dimensional space, $\rho^{d_\mathcal{B}} \in \Gamma \Lambda_\mathbb{H}^{d_\mathcal{B}} \mathcal{K}$ a nowhere vanishing horizontal form, $\psi \in \Gamma \Lambda_\mathbb{V}^0 \mathcal{K}$ a vertical 0-form, and $\tilde{\psi}_{\mathcal{B}_0}$ a functional such that

$$\tilde{\psi}_{\mathcal{B}_0} = \int_{\mathcal{B}_0} \psi \rho^{d_\mathcal{B}}.$$

Then, the total Legendre transform of $\Psi_{\mathcal{B}_0}$ is defined as

$$\tilde{\psi}_{\mathcal{B}_0}^* = \int_{\mathcal{B}_0} \mathfrak{L}[\psi] \rho^{d_\mathcal{B}},$$

where $\mathfrak{L}[\psi]$ is the total Legendre transform of ψ , (see Definition 2.24).

As shown in Propositions 2.4 and 2.5, vector fields on Legendre submanifolds of contact manifolds are concisely written as contact Hamiltonian vector fields with adapted functions introduced in Definition 2.30 for the standard contact geometry. Also, for contact geometry on fiber spaces, similar functions can be defined as follows.

Definition 2.38. (*Adapted functions on fiber space*): Let $(\mathcal{K}, \lambda_\mathbb{V}, \pi, \mathcal{B})$ be a $(2n+1)$ -dimensional contact manifold over a base space \mathcal{B} , \mathcal{C}_ζ a fiber space on $\pi^{-1}(\zeta)$, (x, y, z) canonical coordinates for $\pi^{-1}(\zeta)$, $(\zeta \in \mathcal{B})$ such that $\lambda_\mathbb{V} = d_\mathbb{V} z - y_a d_\mathbb{V} x^a$ with $x = \{x^1, \dots, x^n\}$ and $y = \{y_1, \dots, y_n\}$, and $\mathcal{K} = \bigcup_{\zeta \in \mathcal{B}} \mathcal{C}_\zeta$. In addition let ψ be a vertical function on \mathcal{C}_ζ depending on x , and φ a vertical function on \mathcal{C}_ζ depending on y . Then the functions $\Delta_0^{\zeta\psi}, \{\Delta_1^{\zeta\psi}, \dots, \Delta_n^{\zeta\psi}\} : \mathcal{C}_\zeta \rightarrow \mathbb{R}$, and $\Delta_{\zeta\varphi}^0, \{\Delta_{\zeta\varphi}^1, \dots, \Delta_{\zeta\varphi}^n\} : \mathcal{C}_\zeta \rightarrow \mathbb{R}$ such that

$$\Delta_0^{\zeta\psi}(x, z) := \psi(x) - z, \quad \Delta_a^{\zeta\psi}(x, y) := \frac{\partial \psi}{\partial x^a} - y_a, \quad a \in \{1, \dots, n\}.$$

$$\Delta_{\zeta\varphi}^0(x, y, z) := x^j p_j - \varphi(y) - z, \quad \Delta_{\zeta\varphi}^a(x, y) := x^a - \frac{\partial \varphi}{\partial y_a}, \quad a \in \{1, \dots, n\}.$$

are referred to as adapted functions on the fiber space.

Similar to Proposition 2.3, one has the following.

Proposition 2.8. (*Local expressions of Legendre submanifold with adapted functions*): The Legendre submanifold $\mathcal{A}_{\zeta\psi}$ generated by ψ on $\pi^{-1}(\zeta)$ as (18) is expressed as

$$\mathcal{A}_{\zeta\psi}^c := \Phi_{\mathcal{C}_\zeta \mathcal{A}_{\zeta\psi}} \mathcal{A}_{\zeta\psi} = \left\{ (x, p, z) \in \mathcal{C}_\zeta \mid \Delta_0^{\zeta\psi} = 0 \text{ and } \Delta_1^{\zeta\psi} = \dots = \Delta_n^{\zeta\psi} = 0 \right\}, \quad (20)$$

where $\Phi_{\mathcal{C}_\zeta \mathcal{A}_{\zeta\psi}} : \mathcal{A}_{\zeta\psi} \rightarrow \mathcal{C}_\zeta$ is the embedding. Similarly, the Legendre submanifold $\mathcal{A}_{\zeta\varphi}$ generated by $-\varphi$ as (19) is expressed as

$$\mathcal{A}_{\zeta\varphi}^c := \Phi_{\mathcal{C}_\zeta \mathcal{A}_{\zeta\varphi}} \mathcal{A}_{\zeta\varphi} = \left\{ (x, y, z) \in \mathcal{C}_\zeta \mid \Delta_{\zeta\varphi}^0 = 0 \text{ and } \Delta_{\zeta\varphi}^1 = \dots = \Delta_{\zeta\varphi}^n = 0 \right\}, \quad (21)$$

where $\Phi_{\mathcal{C}_\zeta \mathcal{A}_{\zeta\varphi}} : \mathcal{A}_{\zeta\varphi} \rightarrow \mathcal{C}_\zeta$ is the embedding.

Contact Hamiltonian vertical vector fields are also written in terms of adapted functions on fiber spaces.

Proposition 2.9. (*Restricted contact Hamiltonian vertical vector field as the push-forward of a vector field on the Legendre submanifold generated by ψ*): Let $\{F_{\zeta\psi}^1, \dots, F_{\zeta\psi}^n\}$ be a set of functions of x on $\mathcal{A}_{\zeta\psi}$ such that they do not identically vanish, and $\tilde{X}_{\zeta\psi}^0 \in T_x \mathcal{A}_{\zeta\psi}$, $(x \in \mathcal{A}_{\zeta\psi})$ the vector field given as

$$\tilde{X}_{\zeta\psi}^0 = \dot{x}^a \frac{\partial}{\partial x^a}, \quad \text{where} \quad \dot{x}^a = F_{\zeta\psi}^a(x), \quad (a \in \{1, \dots, n\}).$$

In addition, let $X_{\zeta\psi}^0 := (\Phi_{\mathcal{C}_\zeta\mathcal{A}_{\zeta\psi}})_* \tilde{X}_{\zeta\psi}^0 \in T_\xi \mathcal{A}_{\zeta\psi}^{\mathcal{C}}$, ($\xi \in \mathcal{A}_{\zeta\psi}^{\mathcal{C}}$) be the push-forward of $\tilde{X}_{\zeta\psi}^0$, where $\mathcal{A}_{\zeta\psi}^{\mathcal{C}} := \Phi_{\mathcal{C}_\zeta\mathcal{A}_{\zeta\psi}} \mathcal{A}_{\zeta\psi}$ with $\Phi_{\mathcal{C}_\zeta\mathcal{A}_{\zeta\psi}} : \mathcal{A}_{\zeta\psi} \rightarrow \mathcal{C}_\zeta$ being the embedding :

$$\begin{aligned} \Phi_{\mathcal{C}_\zeta\mathcal{A}_{\zeta\psi}} &: \mathcal{A}_{\zeta\psi} \rightarrow \mathcal{A}_{\zeta\psi}^{\mathcal{C}}, & x &\mapsto (x, y(x), z(x)), & \text{on } \pi^{-1}(\zeta), \\ (\Phi_{\mathcal{C}_\zeta\mathcal{A}_{\zeta\psi}})_* &: T_x \mathcal{A}_{\zeta\psi} \rightarrow T_\xi \mathcal{A}_{\zeta\psi}^{\mathcal{C}}, & \tilde{X}_{\zeta\psi}^0 &\mapsto X_{\zeta\psi}^0. \end{aligned}$$

Then it follows that

$$X_\psi^0 = \dot{x}^a \frac{\partial}{\partial x^a} + \dot{y}_a \frac{\partial}{\partial y_a} + \dot{z} \frac{\partial}{\partial z}, \quad \text{where } \dot{x}^a = F_{\zeta\psi}^a(x), \quad \dot{y}_a = \frac{d}{dt} \left(\frac{\partial \psi}{\partial x^a} \right), \quad \dot{z} = \frac{d\psi}{dt}. \quad (22)$$

In addition, one has that $X_{\zeta\psi}^0 = X_{\tilde{h}_\psi}^0|_{\tilde{h}_\psi=0}$. Here $X_{\tilde{h}_\psi}$ is the contact Hamiltonian vertical vector field associated with

$$h_\psi(x, y, z) = \Delta_0^{\zeta\psi}(x, y) F_{\zeta\psi}^a(x) + \Gamma_{\zeta\psi}(\Delta_0^{\zeta\psi}(x, z)), \quad (23)$$

where $\Gamma_{\zeta\psi}$ is a function of $\Delta_0^{\zeta\psi}$ such that

$$\Gamma_{\zeta\psi}(\Delta_0^{\zeta\psi}) = \begin{cases} 0 & \text{for } \Delta_0^{\zeta\psi} = 0 \\ \text{non-zero} & \text{for } \Delta_0^{\zeta\psi} \neq 0 \end{cases}.$$

There exists a counterpart of Proposition 2.9, that is given as follows.

Proposition 2.10. (Restricted contact Hamiltonian vertical vector field as the push-forward of a vector field on the Legendre submanifold generated by $-\varphi$): Let $\{F_{\zeta,1}^\varphi, \dots, F_{\zeta,n}^\varphi\}$ be a set of functions of y on $\mathcal{A}_{\zeta\varphi}$ such that they do not identically vanish, and $\tilde{X}_0^{\zeta\varphi} \in T_y \mathcal{A}_{\zeta\varphi}$, ($y \in \mathcal{A}_{\zeta\varphi}$) the vector field given as

$$\tilde{X}_0^{\zeta\varphi} = \dot{y}_a \frac{\partial}{\partial y_a}, \quad \text{where } \dot{y}_a = F_a^{\zeta\varphi}(y), \quad (a \in \{1, \dots, n\}).$$

In addition, let $X_0^{\zeta\varphi} := (\Phi_{\mathcal{C}_\zeta\mathcal{A}_{\zeta\varphi}})_* \tilde{X}_0^{\zeta\varphi} \in T_\xi \mathcal{A}_{\zeta\varphi}^{\mathcal{C}}$, ($\xi \in \mathcal{A}_{\zeta\varphi}^{\mathcal{C}}$) be the push-forward of $\tilde{X}_0^{\zeta\varphi}$, where $\mathcal{A}_{\zeta\varphi}^{\mathcal{C}} := \Phi_{\mathcal{C}_\zeta\mathcal{A}_{\zeta\varphi}} \mathcal{A}_{\zeta\varphi}$ with $\Phi_{\mathcal{C}_\zeta\mathcal{A}_{\zeta\varphi}} : \mathcal{A}_{\zeta\varphi} \rightarrow \mathcal{C}_\zeta$ being the embedding :

$$\begin{aligned} \Phi_{\mathcal{C}_\zeta\mathcal{A}_{\zeta\varphi}} &: \mathcal{A}_{\zeta\varphi} \rightarrow \mathcal{A}_{\zeta\varphi}^{\mathcal{C}}, & y &\mapsto (x(y), y, z(y)), & \text{on } \pi^{-1}(\zeta), \\ (\Phi_{\mathcal{C}_\zeta\mathcal{A}_{\zeta\varphi}})_* &: T_y \mathcal{A}_{\zeta\varphi} \rightarrow T_\xi \mathcal{A}_{\zeta\varphi}^{\mathcal{C}}, & \tilde{X}_0^{\zeta\varphi} &\mapsto X_0^{\zeta\varphi}. \end{aligned}$$

Then, it follows that

$$X_0^{\zeta\varphi} = \dot{x}^a \frac{\partial}{\partial x^a} + \dot{y}_a \frac{\partial}{\partial y_a} + \dot{z} \frac{\partial}{\partial z}, \quad \text{where } \dot{x}^a = \frac{d}{dt} \left(\frac{\partial \varphi}{\partial y_a} \right), \quad \dot{y}_a = F_a^{\zeta\varphi}(y), \quad \dot{z} = y_j F_k^{\zeta\varphi} \frac{\partial^2 \varphi}{\partial y_k \partial y_j}. \quad (24)$$

In addition, one has that $X_0^{\zeta\varphi} = X_{\tilde{h}_\varphi}^0|_{\tilde{h}_\varphi=0}$. Here $X_{\tilde{h}_\varphi}$ is the contact Hamiltonian vertical vector field associated with

$$h_\varphi(x, y, z) = \Delta_{\zeta\varphi}^a(x, y) F_a^{\zeta\varphi}(y) + \Gamma^{\zeta\varphi}(\Delta_{\zeta\varphi}^0(x, y, z)), \quad (25)$$

where $\Gamma^{\zeta\varphi}$ is a function of $\Delta_{\zeta\varphi}^0$ such that

$$\Gamma^{\zeta\varphi}(\Delta_{\zeta\varphi}^0) = \begin{cases} 0 & \text{for } \Delta_{\zeta\varphi}^0 = 0 \\ \text{non-zero} & \text{for } \Delta_{\zeta\varphi}^0 \neq 0 \end{cases}.$$

2.6 Information geometry

Information geometry is a geometrization of parametric statistics[18], and it was shown that there are some overlap between contact geometry and information geometry[6] (see also Refs.[5, 7]). Later on in this paper it will be shown that a class of Hamiltonian functionals for distributed-parameter port-Hamiltonian systems with respect to Stokes-Dirac structures can induce information geometry. To this end, definitions and known theorems are summarized below. The following definitions follow the standard information geometry.

Definition 2.39. (Affine-coordinate and flat connection, [18]): Let \mathcal{M} be an n -dimensional manifold, $x = \{x^1, \dots, x^n\}$ coordinates, ∇ a connection, $\{\Gamma_{ab}^c\}$ connection coefficients such that $\nabla_{\partial_a} \partial_b = \Gamma_{ab}^c \partial_c$, ($\partial_a := \partial/\partial x^a$). If $\{\Gamma_{ab}^c\} \equiv 0$ hold for all $\xi \in \mathcal{M}$, then x is referred to as a ∇ -affine coordinate system, or affine coordinates. If it is the case, then ∇ is referred to as a flat connection.

Definition 2.40. (Dual connection, [18]): Let (\mathcal{M}, g) be an n -dimensional Riemannian or pseudo-Riemannian manifold, and ∇ a connection. If a connection ∇^* satisfies

$$Z[g(X, Y)] = g(\nabla_Z X, Y) + g(X, \nabla_Z^* Y), \quad \forall X, Y, Z \in \Gamma T\mathcal{M} \quad (26)$$

then ∇ and ∇^* are referred to as dual connections, also ∇^* is referred to as a dual connection of ∇ with respect to g .

Lemma 2.4. (Existence of a unique dual connection, [18]): Given a metric tensor field and a connection, there exists a unique dual connection.

Definition 2.41. (Dual coordinates, [18]): Let (\mathcal{M}, g) be an n -dimensional Riemannian or pseudo-Riemannian manifold, $x = \{x^1, \dots, x^n\}$ a set of local coordinates, and $y = \{y_1, \dots, y_n\}$ another set of local coordinates. If

$$g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y_b}\right) = \delta_a^b, \quad (27)$$

then y is referred to as the dual coordinate system. If it is the case, then x and y are referred to as being mutually dual with respect to g .

Combining Definitions 2.39, 2.40 and 2.41, one has the following.

Lemma 2.5. (Dual coordinates and affine coordinates) : Let (\mathcal{M}, g) be an n -dimensional Riemannian or pseudo-Riemannian manifold, ∇ a connection, $x = \{x^1, \dots, x^n\}$ a set of ∇ -affine coordinates, and $y = \{y_1, \dots, y_n\}$ another set of coordinates. If x and y are mutually dual with respect to g , then y is a ∇^* -affine coordinate system.

The following space plays various roles in information geometry.

Definition 2.42. (Dually flat space, [18]): Let (\mathcal{M}, g) be an n -dimensional Riemannian or pseudo-Riemannian manifold, ∇ and ∇^* dual connections. If there exist ∇ -affine coordinates and ∇^* -affine ones, then $(\mathcal{M}, g, \nabla, \nabla^*)$ is referred to as a dually flat space.

From these definitions, one can show the following relation between pairings and inner products.

Proposition 2.11. (Inner products and pairings on a dually flat space): Let $(\mathcal{M}, g, \nabla, \nabla^*)$ be an n -dimensional dually flat space, $x = \{x^1, \dots, x^n\}$ a set of ∇ -affine coordinates, $y = \{y_1, \dots, y_n\}$ a set of ∇^* -affine coordinates. If the inner products $T_\xi \mathcal{M} \times T_\xi \mathcal{M} \rightarrow \mathbb{R}, (\xi \in \mathcal{M})$ between the bases $\{\partial/\partial x^1, \dots, \partial/\partial x^n\}$ and $\{\partial/\partial y_1, \dots, \partial/\partial y_n\}$ are given as

$$\begin{aligned} g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) &= g_{ab}, & g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y_b}\right) &= \delta_a^b, \\ g\left(\frac{\partial}{\partial y_a}, \frac{\partial}{\partial x^b}\right) &= \delta_b^a, & g\left(\frac{\partial}{\partial y_a}, \frac{\partial}{\partial y_b}\right) &= g^{ab}, \end{aligned}$$

(i.e., x and y are mutually dual with respect to g), then one has the following pairings $T_\xi^* \mathcal{M} \times T_\xi \mathcal{M} \rightarrow \mathbb{R}, (\xi \in \mathcal{M})$

$$\begin{aligned} dx^a \left(\frac{\partial}{\partial x^b} \right) &= \delta^{ab}, & dx^a \left(\frac{\partial}{\partial y_b} \right) &= g^{ab}, \\ dy_a \left(\frac{\partial}{\partial x^b} \right) &= g_{ab}, & dy_a \left(\frac{\partial}{\partial y_b} \right) &= \delta_{ab}. \end{aligned}$$

As shown in Ref.[6], a contact manifold and a strictly convex function induce a dually flat space.

Theorem 2.9. (Contact manifold and a function induce a dually flat space, [6]) : Let (\mathcal{C}, λ) be a $(2n+1)$ -dimensional contact manifold, (x, y, z) the canonical coordinates such that $\lambda = dz - y_a dx^a$ with $x = \{x^1, \dots, x^n\}$ and $y = \{y_1, \dots, y_n\}$, and ψ a strictly convex function of x only. If the Legendre submanifold generated by ψ is simply connected, then $((\mathcal{C}, \lambda), \psi)$ induces an n -dimensional dually flat space $(\Phi_{\mathcal{C}\mathcal{A}\psi} \mathcal{A}_\psi, g, \nabla, \nabla^*)$ with $\Phi_{\mathcal{C}\mathcal{A}\psi} : \mathcal{A}_\psi \rightarrow \mathcal{C}$ being the embedding.

The following is often used in information geometry.

Definition 2.43. (Canonical divergence, [18]): Let $(\mathcal{M}, g, \nabla, \nabla^*)$ be an n -dimensional dually flat space, $\{x^1, \dots, x^n\}$ ∇ -affine coordinates, $\{y_1, \dots, y_n\}$ ∇^* -affine coordinates, ξ and ξ' two points of \mathcal{M} . Then, the function $\mathbb{D} : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$,

$$\mathbb{D}(\xi \parallel \xi') := \psi(\xi) + \varphi(\xi') - x^a|_\xi y_a|_{\xi'}, \quad (28)$$

is referred to as the canonical divergence.

Remark 2.15. There is another convention for the canonical divergence (see Ref. [19]).

In information geometry, the following theorem is well-known.

Theorem 2.10. (Generalized Pythagorean theorem, [18]): Let $(\mathcal{M}, g, \nabla, \nabla^*)$ be a dually flat space, $\mathbb{D} : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ the canonical divergence, ξ', ξ'', ξ''' be three points of \mathcal{M} such that 1. ξ' and ξ'' are connected with the ∇^* -geodesic curve and 2. ξ'' and ξ''' are connected with the ∇ -geodesic curve. Then, it follows that

$$\mathbb{D}(\xi''' \parallel \xi') = \mathbb{D}(\xi''' \parallel \xi'') + \mathbb{D}(\xi'' \parallel \xi').$$

2.7 Stokes-Dirac structure

To discuss the Hamiltonian formulation of distributed-parameter systems on bounded spatial domain, Dirac structure was extended. That extended structure is referred to as Stokes-Dirac structure[27]. After some spaces are introduced, the definition of Stokes-Dirac structure and a fundamental theorem are given. Also some definitions are given so that distributed-parameter systems can be discussed in the following sections.

Definition 2.44. (Spaces of flow variables and effort variables): Let \mathcal{Z} be an n -dimensional connected manifold, and p, q natural numbers satisfying $0 \leq p, q \leq n$ and $p + q = n + 1$. Then

$$\mathcal{F}_{p,q} := \Gamma \Lambda^p \mathcal{Z} \times \Gamma \Lambda^q \mathcal{Z} \times \Gamma \Lambda^{n-p} \partial \mathcal{Z}, \quad \mathcal{E}_{p,q} := \Gamma \Lambda^{n-p} \mathcal{Z} \times \Gamma \Lambda^{n-q} \mathcal{Z} \times \Gamma \Lambda^{n-q} \partial \mathcal{Z}, \quad (29)$$

are referred to as the space of flow variables on \mathcal{Z} , and the space of effort variables on \mathcal{Z} , respectively. In addition, elements of $\mathcal{F}_{p,q}$ and those of $\mathcal{E}_{p,q}$ are referred to as flow variables and effort variables, respectively.

Definition 2.45. (Bilinear form for Stokes-Dirac structure, [27]): Let \mathcal{Z} be an n -dimensional manifold, $\mathcal{F}_{p,q}$ the space of flow variables, $\mathcal{E}_{p,s}$ the space of effort variables. Then, the map $\langle\langle -, - \rangle\rangle : \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \rightarrow \mathbb{R}$ given by

$$\begin{aligned} & \langle\langle (\mathbf{f}_p, \mathbf{f}_q, \mathbf{f}_\partial, \mathbf{e}_p, \mathbf{e}_q, \mathbf{e}_\partial), (\mathbf{f}'_p, \mathbf{f}'_q, \mathbf{f}'_\partial, \mathbf{e}'_p, \mathbf{e}'_q, \mathbf{e}'_\partial) \rangle\rangle \\ &= \int_{\mathcal{Z}} \mathbf{e}_p \wedge \mathbf{f}'_p + \mathbf{e}_q \wedge \mathbf{f}'_q + \mathbf{e}'_p \wedge \mathbf{f}_p + \mathbf{e}'_q \wedge \mathbf{f}_q + \int_{\partial \mathcal{Z}} \mathbf{e}_\partial \wedge \mathbf{f}'_\partial + \mathbf{e}'_\partial \wedge \mathbf{f}_\partial, \end{aligned} \quad (30)$$

for

$$\mathbf{f}_p, \mathbf{f}'_p \in \Gamma \Lambda^p \mathcal{Z}, \quad \mathbf{f}_q, \mathbf{f}'_q \in \Gamma \Lambda^q \mathcal{Z}, \quad \mathbf{e}_p, \mathbf{e}'_p \in \Gamma \Lambda^{n-p} \mathcal{Z}, \quad \mathbf{e}_q, \mathbf{e}'_q \in \Gamma \Lambda^{n-q} \mathcal{Z},$$

and

$$\mathbf{f}_\partial, \mathbf{f}'_\partial \in \Gamma \Lambda^{n-p} \partial \mathcal{Z}, \quad \mathbf{e}_\partial, \mathbf{e}'_\partial \in \Gamma \Lambda^{n-q} \partial \mathcal{Z},$$

Then the following theorem holds.

Theorem 2.11. (*Stokes-Dirac structure, [27]*): Let \mathcal{Z} be an n -dimensional connected manifold, $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$ the space of flow variables on \mathcal{Z} and that of effort variables on \mathcal{Z} given in (29) with p, q satisfying $0 \leq p, q \leq n$ and $p + q = n + 1$. Define the relations

$$\begin{pmatrix} \mathbf{f}_p \\ \mathbf{f}_q \end{pmatrix} = \begin{pmatrix} 0 & (-1)^r d \\ d & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_p \\ \mathbf{e}_q \end{pmatrix}, \quad \begin{pmatrix} \mathbf{f}_\partial \\ \mathbf{e}_\partial \end{pmatrix} = \begin{pmatrix} \mathbf{e}_p|_{\partial\mathcal{Z}} \\ -(-1)^{n-q} \mathbf{e}_q|_{\partial\mathcal{Z}} \end{pmatrix}, \quad r = pq + 1. \quad (31)$$

Then, the following linear subspace $\mathbf{D}_{\mathcal{Z}}$ of $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$

$$\mathbf{D}_{\mathcal{Z}} := \left\{ (\mathbf{f}_p, \mathbf{f}_q, \mathbf{f}_\partial, \mathbf{e}_p, \mathbf{e}_q, \mathbf{e}_\partial) \mid \text{The set of equations (31) hold.} \right\}, \quad (32)$$

satisfies $\mathbf{D}_{\mathcal{Z}} = \mathbf{D}_{\mathcal{Z}}^\perp$, with \perp denoting the orthogonal complement with respect to the bilinear form given by (30).

Definition 2.46. The linear subspace $\mathbf{D}_{\mathcal{Z}}$ in Theorem 2.11 is referred to as a Stokes-Dirac structure.

This paper studies the following class of Stokes-Dirac structures.

Definition 2.47. (*Distributed-parameter port-Hamiltonian system, [27]*): Let \mathcal{Z} be an n -dimensional manifold, $\tilde{\psi} \in \Gamma F\mathcal{Z}$ a functional, $\Gamma\Lambda^p\mathcal{Z} \times \Gamma\Lambda^q\mathcal{Z}$ a state space with $p + q = n + 1$ and $0 \leq p, q \leq n$, and $\mathbf{D}_{\mathcal{Z}}$ the Stokes-Dirac structure given by (32). Then, for $(\boldsymbol{\alpha}_p, \boldsymbol{\alpha}_q) \in \Gamma\Lambda^p\mathcal{Z} \times \Gamma\Lambda^q\mathcal{Z}$ with $r = pq + 1$, identify

$$\mathbf{f}_p = -\frac{\partial}{\partial t} \boldsymbol{\alpha}_p, \quad \mathbf{f}_q = -\frac{\partial}{\partial t} \boldsymbol{\alpha}_q, \quad \mathbf{e}_p = \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_p}, \quad \mathbf{e}_q = \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_q}, \quad (33)$$

such that

$$\begin{pmatrix} -\frac{\partial}{\partial t} \boldsymbol{\alpha}_p \\ -\frac{\partial}{\partial t} \boldsymbol{\alpha}_q \end{pmatrix} = \begin{pmatrix} 0 & (-1)^r d \\ d & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta}{\delta \boldsymbol{\alpha}_p} \tilde{\psi} \\ \frac{\delta}{\delta \boldsymbol{\alpha}_q} \tilde{\psi} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{f}_\partial \\ \mathbf{e}_\partial \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{pmatrix} \begin{pmatrix} \frac{\delta}{\delta \boldsymbol{\alpha}_p} \tilde{\psi} \\ \frac{\delta}{\delta \boldsymbol{\alpha}_q} \tilde{\psi} \end{pmatrix} \Big|_{\partial\mathcal{Z}}.$$

This system is referred to as the distributed-parameter port-Hamiltonian systems with \mathcal{Z} .

Remark 2.16. By the power-conserving property, it follows for any $(\mathbf{f}_p, \mathbf{f}_q, \mathbf{f}_\partial, \mathbf{e}_p, \mathbf{e}_q, \mathbf{e}_\partial)$ in the Stokes-Dirac structure that

$$\int_{\mathcal{Z}} (\mathbf{e}_p \wedge \mathbf{f}_p + \mathbf{e}_q \wedge \mathbf{f}_q) + \int_{\partial\mathcal{Z}} \mathbf{e}_\partial \wedge \mathbf{f}_\partial = 0. \quad (34)$$

Thus, one has from (33) and (34) that

$$\frac{d}{dt} \tilde{\psi} = \int_{\mathcal{Z}} \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_p} \wedge \frac{\partial \boldsymbol{\alpha}_p}{\partial t} + \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_q} \wedge \frac{\partial \boldsymbol{\alpha}_q}{\partial t} = - \int_{\mathcal{Z}} (\mathbf{e}_p \wedge \mathbf{f}_p + \mathbf{e}_q \wedge \mathbf{f}_q) = \int_{\partial\mathcal{Z}} \mathbf{e}_\partial \wedge \mathbf{f}_\partial.$$

Physically the manifold \mathcal{Z} is used for expressing spatial domain. Thus, the cases $n = 1, 2$, and 3 are focused in this paper.

Examples of distributed-parameter port-Hamiltonian systems have been given in Ref. [27]. They are Maxwell's equations, telegraph equation, vibrating string, ideal fluid. Furthermore, various models can be described such as shallow water equations[25].

Definition 2.48. (*Energy functional and energy density function*): A functional $\tilde{\psi} \in \Gamma F\mathcal{Z}$ used in Definition 2.47 that can depend on $\boldsymbol{\alpha}_p \in \Gamma\Lambda^p\mathcal{Z}$ and $\boldsymbol{\alpha}_q \in \Gamma\Lambda^q\mathcal{Z}$ is referred to as an energy functional. On a Riemannian manifold (\mathcal{Z}, g) with g being a Riemannian metric tensor field, let $\star 1$ be a canonical volume-form. If $\tilde{\psi}$ can be written as

$$\tilde{\psi} = \int_{\mathcal{Z}} \psi \star 1,$$

with $\psi \in \Gamma\Lambda^0\mathcal{Z}$ being a function, then ψ is referred to as an energy density function.

Definition 2.49. (*Energy mixed form*): Let $\tilde{\psi}$ be an energy functional for a Stokes-Dirac structure. Then $\boldsymbol{\alpha}_p$ and $\boldsymbol{\alpha}_q$ are referred to as energy mixed forms.

Definition 2.50. (*Co-energy mixed forms*): Let $\tilde{\psi}$ be an energy functional for a Stokes-Dirac structure. Then $\mathbf{e}_p \in \Gamma\Lambda^{n-p}\mathcal{Z}$ and $\mathbf{e}_q \in \Gamma\Lambda^{n-q}\mathcal{Z}$ derived from $\tilde{\psi}$, as appeared in (33) as

$$\mathbf{e}_p = \frac{\delta\tilde{\psi}}{\delta\boldsymbol{\alpha}_p}, \quad \text{and} \quad \mathbf{e}_q = \frac{\delta\tilde{\psi}}{\delta\boldsymbol{\alpha}_q},$$

are referred to as co-energy mixed forms.

The relations between $\mathbf{e}_p, \mathbf{e}_q$ and $\boldsymbol{\alpha}_p, \boldsymbol{\alpha}_q$ in Definition 2.50 can be seen as constitutive relations. For example, for the case of $(3+1)$ decomposed Maxwell's equations on a Riemannian manifold, these constitutive relations are $\mathbf{e} = \varepsilon^{-1} \star \mathbf{D}, \mathbf{h} = \mu^{-1} \star \mathbf{B}$ where \mathbf{D} is a 2-form displacement field, \mathbf{B} a 2-form magnetic induction field, \mathbf{e} a 1-form electric field, \mathbf{h} a 1-form magnetic field, ε a permittivity, and μ a permeability (see Ref. [17]).

Given a Riemannian manifold, the following energy functionals are mainly focused in this paper.

Definition 2.51. (*Quadratic energy functional*): Let (\mathcal{Z}, g) be a Riemannian manifold with g being a Riemannian metric field, and \star a canonical volume form. If $\tilde{\psi}$ is quadratic in the sense that

$$\tilde{\psi}[\boldsymbol{\alpha}_p, \boldsymbol{\alpha}_q] = \frac{1}{2} \int_{\mathcal{Z}} (\boldsymbol{\alpha}_p \wedge \star \boldsymbol{\alpha}_p + \boldsymbol{\alpha}_q \wedge \star \boldsymbol{\alpha}_q), \quad (35)$$

then $\tilde{\psi}$ is referred to as a quadratic energy functional.

Remark 2.17. If $\tilde{\psi}$ is quadratic, then it follows that

$$\mathbf{e}_p = \star \boldsymbol{\alpha}_p, \quad \text{and} \quad \mathbf{e}_q = \star \boldsymbol{\alpha}_q.$$

For the energy functional $\tilde{\psi}$ given in (35), one can introduce the co-energy functional.

Definition 2.52. (*Co-energy functional*): Given an energy functional, its total Legendre transform of functional (see Definition 2.37) is referred to as a co-energy functional.

In this paper the following class of co-energy functionals is focused.

Definition 2.53. (*Quadratic co-energy functional*): Given a functional $\tilde{\psi}$ in (35), the functional

$$\tilde{\varphi}[\mathbf{e}_p, \mathbf{e}_q] = \frac{1}{2} \int_{\mathcal{Z}} (\mathbf{e}_p \wedge \star \mathbf{e}_p + \mathbf{e}_q \wedge \star \mathbf{e}_q), \quad \mathbf{e}_p = \frac{\delta\tilde{\psi}}{\delta\boldsymbol{\alpha}_p}, \quad \mathbf{e}_q = \frac{\delta\tilde{\psi}}{\delta\boldsymbol{\alpha}_q}, \quad (36)$$

is referred to as the co-energy functional.

The functional $\tilde{\psi}$ in (35) depends on $\boldsymbol{\alpha}_p$ and $\boldsymbol{\alpha}_q$. On the other hand the functional $\tilde{\varphi}$ in (36) depends on the effort variables, \mathbf{e}_p and \mathbf{e}_q . In Section 4, it will be shown that this $\tilde{\varphi}$ is the total Legendre transform of the $\tilde{\psi}$ in the sense of Definition 2.37 (see Proposition 4.1).

3 Systems with respect to Stokes-Dirac structures described as contact Hamiltonian vector fields

In this section it is shown that a class of distributed-parameter port-Hamiltonian systems with respect to Stokes-Dirac structures are written in terms of contact geometry, where such systems are assumed to be described on Riemannian manifolds. To this end, contact bundles are used where base spaces are Riemannian manifolds.

To describe distributed-parameter port-Hamiltonian systems in a contact geometric language, one defines the following.

Definition 3.1. (*Adapted mixed forms*): Let (\mathcal{Z}, g) be an n -dimensional Riemannian manifold, $(\mathcal{K}, \lambda_{\mathcal{V}}, \pi, \mathcal{Z})$ a contact manifold over \mathcal{Z} , p and q natural numbers satisfying $0 \leq p, q \leq n$ and $p + q = n + 1$, \mathcal{C}_{ζ} a $(2({}_nC_p + {}_nC_q) + 1)$ -dimensional contact manifold over a point $\zeta \in \mathcal{Z}$ such that $\mathcal{K} = \bigcup_{\zeta \in \mathcal{B}} \mathcal{C}_{\zeta}$ with

${}_nC_p = n!/(p!(n-p)!)$, $\tilde{\psi} \in \Gamma FK$ an energy functional consisting of $\alpha_p \in \Gamma \Lambda_{\mathbb{H},\mathbb{V}}^{p,0} \mathcal{K}$ and $\alpha_q \in \Gamma \Lambda_{\mathbb{H},\mathbb{V}}^{q,0} \mathcal{K}$, ψ an energy density function on \mathcal{Z} , $e_p \in \Gamma \Lambda_{\mathbb{H},\mathbb{V}}^{n-p,0} \mathcal{K}$, $e_q \in \Gamma \Lambda_{\mathbb{H},\mathbb{V}}^{n-q,0} \mathcal{K}$, and \mathcal{E} the coordinate such that the Reeb vertical vector field is $\partial/\partial \mathcal{E}$. Then

$$\Delta_0^{\zeta\psi} := \psi - \mathcal{E}, \quad \Delta_p^{\zeta\psi} := \frac{\delta \tilde{\psi}}{\delta \alpha_p} - e_p, \quad \Delta_q^{\zeta\psi} := \frac{\delta \tilde{\psi}}{\delta \alpha_q} - e_q, \quad (1 \leq p, q \leq n) \quad (37)$$

are referred to as adapted mixed forms.

Remark 3.1. The dimension of the space for α_p is ${}_nC_p$, which is the same as that for e_p due to ${}_nC_{n-p} = {}_nC_p$. Similarly the dimension of the space for α_q is ${}_nC_q$, which is the same as that for e_q due to ${}_nC_{n-q} = {}_nC_q$. The explicit correspondences between canonical coordinates (x, y, z) for \mathcal{C}_ζ and $\alpha_p, \alpha_q, e_p, e_q$ are discussed after n, p, q are fixed.

With the adapted mixed forms, the distributed-parameter port-Hamiltonian systems can be formulated in the following space.

Definition 3.2. (Phase space for distributed-parameter port-Hamiltonian system): Let $\mathcal{A}_{\zeta\psi}^{\mathcal{C}}$ be the Legendre submanifold of the vertical space generated by ψ as

$$\mathcal{A}_{\zeta\psi}^{\mathcal{C}} = \left\{ (x, y, z) \in \pi^{-1}(\zeta) \mid \Delta_0^{\zeta\psi} = \Delta_p^{\zeta\psi} = \Delta_q^{\zeta\psi} = 0 \right\}. \quad (38)$$

Then the sub-bundle $(\mathcal{A}_{\zeta\psi}^{\mathcal{K}}, \pi|_{\mathcal{A}_{\zeta\psi}^{\mathcal{K}}}, \mathcal{Z})$ with $\mathcal{A}_{\zeta\psi}^{\mathcal{K}} = \bigcup_{\zeta \in \mathcal{B}} \mathcal{A}_{\zeta\psi}^{\mathcal{C}}$ is referred to as the phase space for the distributed-parameter port-Hamiltonian systems.

3.1 Tree-dimensional Riemannian manifold

Before stating the main theorems in this paper, we note the following. On the phase space for the distributed-parameter port-Hamiltonian systems, one has the distributed-parameter port-Hamiltonian systems. For the case of $n = 3$, the possible combinations of $0 \leq p, q \leq n$ that satisfy $p + q = n + 1$ are

- $\{p = 1, q = 3\}, \{p = 3, q = 1\},$
- $\{p = q = 2\}.$

Since the case $\{p = 3, q = 1\}$ can reduce to $\{p = 1, q = 3\}$, attention is concentrated on the cases $\{p = 1, q = 3\}$ and $\{p = q = 2\}$. Taking into account these combinations, one has the following one of main theorems.

Theorem 3.1. (Distributed-parameter port-Hamiltonian system as contact Hamiltonian vertical vector field): Let (\mathcal{Z}, g) be a connected 3-dimensional Riemannian manifold, and ψ an energy density function specified later.

- For the case $\{p = 1, q = 3\}$, let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{Z})$ be a 9-dimensional contact manifold over the base space \mathcal{Z} with $\lambda_{\mathbb{V}} = d_{\mathbb{V}}z - p_a d_{\mathbb{V}}x^a$, $\{\sigma^a\} \in \Gamma \Lambda_{\mathbb{H}}^1 \mathcal{K}$ an orthonormal co-frame,

$$\alpha_p = (\alpha_p)_a \sigma^a \in \Gamma \Lambda_{\mathbb{H},\mathbb{V}}^{1,0} \mathcal{K}, \quad \alpha_q = (\alpha_q)_0 \star 1 \in \Gamma \Lambda_{\mathbb{H},\mathbb{V}}^{3,0} \mathcal{K},$$

$$e_p = \frac{1}{2} (e_p)_{ab} \sigma^a \wedge \sigma^b \in \Gamma \Lambda_{\mathbb{H},\mathbb{V}}^{2,0} \mathcal{K}, \quad e_q = (e_q)_0 \in \Gamma \Lambda_{\mathbb{H},\mathbb{V}}^{0,0} \mathcal{K}, \quad \text{with } (e_p)_{ba} = -(e_p)_{ab}$$

mixed forms, (x, y, z) canonical coordinates with $x = \{x_{(p)}, x_{(q)}\}$, $y = \{y^{(p)}, y^{(q)}\}$,

$$\begin{aligned} x &= \{(\alpha_p)^1, (\alpha_p)^2, (\alpha_p)^3, \star \alpha_q\}, & x_{(p)}^a &= \delta^{ab} (\alpha_p)_b, & x_{(q)} &= \star \alpha_q = (\alpha_q)_0, \\ y &= \{(\star e_p)_1, (\star e_p)_2, (\star e_p)_3, e_q\}, & y_a^{(p)} &= (\star e_p)_a, & y^{(q)} &= (e_q)_0, & \star e_p &= (\star e_p)_a \sigma^a, \\ z &= \mathcal{E}, \end{aligned}$$

where \mathcal{E} is the value of the energy density such that $\mathcal{E} = \psi$ on $\mathcal{A}_{\zeta\psi}^{\mathcal{C}}$ with ψ depending only on x .

- For the case $\{p = q = 2\}$, let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{Z})$ be a 13-dimensional contact manifold over the base space \mathcal{Z} with $\lambda_{\mathbb{V}} = \mathrm{d}_{\mathbb{V}}z - p_a \mathrm{d}_{\mathbb{V}}x^a$, $\{\sigma^a\} \in \Gamma\Lambda_{\mathbb{H}}^1\mathcal{K}$ an orthonormal co-frame,

$$\alpha_p = \frac{1}{2}(\alpha_p)_{ab} \sigma^a \wedge \sigma^b \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{2,0}\mathcal{K}, \quad \alpha_q = \frac{1}{2}(\alpha_q)_{ab} \sigma^a \wedge \sigma^b \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{2,0}\mathcal{K},$$

with $(\alpha_p)_{ba} = -(\alpha_p)_{ab}$, $(\alpha_q)_{ba} = -(\alpha_q)_{ab}$,

$$e_p = (e_p)_a \sigma^a \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{1,0}\mathcal{K}, \quad e_q = (e_q)_a \sigma^a \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{1,0}\mathcal{K}$$

mixed forms, (x, y, z) be canonical coordinates with $x = \{x_{(p)}, x_{(q)}\}$, $y = \{y^{(p)}, y^{(q)}\}$,

$$\begin{aligned} x &= \{(\star\alpha_p)^1, (\star\alpha_p)^2, (\star\alpha_p)^3, (\star\alpha_q)^1, (\star\alpha_q)^2, (\star\alpha_q)^3\}, \\ x_{(p)}^a &= \delta^{ab}(\star\alpha_p)_b, \quad x_{(q)}^a = \delta^{ab}(\star\alpha_q)_b, \\ y &= \{(e_p)_1, (e_p)_2, (e_p)_3, (e_q)_1, (e_q)_2, (e_q)_3\}, \quad y_a^{(p)} = (e_p)_a, \quad y_a^{(q)} = (e_q)_a, \\ z &= \mathcal{E}, \end{aligned}$$

where \mathcal{E} is the value of the energy density such that $\mathcal{E} = \psi$ on $\mathcal{A}_{\zeta\psi}^C$ with ψ depending only on x .

For the both of cases, let $\tilde{\psi}$ be a quadratic energy functional such that

$$\tilde{\psi} = \frac{1}{2} \int_{\mathcal{Z}} (\alpha_p \wedge \star\alpha_p + \alpha_q \wedge \star\alpha_q) = \int_{\mathcal{Z}} \psi \star 1, \quad \text{so that} \quad e_p = \frac{\delta\tilde{\psi}}{\delta\alpha_p} \quad \text{and} \quad e_q = \frac{\delta\tilde{\psi}}{\delta\alpha_q}.$$

In addition, choose the contact Hamiltonian functional as

$$\tilde{h}_{\psi} = \int_{\mathcal{Z}} h_{\psi} \star 1 = \int_{\mathcal{Z}} \left[\Delta_p^{\zeta\psi} \wedge (-F_{\psi}^{\zeta p}) + \Delta_q^{\zeta\psi} \wedge (-F_{\psi}^{\zeta q}) + \Gamma_{\zeta\psi} \left(\Delta_0^{\zeta\psi} \right) \star 1 \right],$$

where $\Delta_0^{\zeta\psi}, \Delta_p^{\zeta\psi}, \Delta_q^{\zeta\psi}$ are defined in (37), $h_{\psi} \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{0,0}\mathcal{K}$, $F_{\psi}^{\zeta p} \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{p,0}\mathcal{K}$, $F_{\psi}^{\zeta q} \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{q,0}\mathcal{K}$ are

$$h_{\psi} = \star^{-1} \left[\Delta_p^{\zeta\psi} \wedge (-F_{\psi}^{\zeta p}) \right] + \star^{-1} \left[\Delta_q^{\zeta\psi} \wedge (-F_{\psi}^{\zeta q}) \right] + \Gamma_{\zeta\psi} \left(\Delta_0^{\zeta\psi} \right),$$

$$F_{\psi}^{\zeta p} := (-1)^r \mathrm{d} \left(\frac{\delta\tilde{\psi}}{\delta\alpha_q} \right) = (-1)^r \mathrm{d} \star\alpha_q, \quad F_{\psi}^{\zeta q} := \mathrm{d} \left(\frac{\delta\tilde{\psi}}{\delta\alpha_p} \right) = \mathrm{d} \star\alpha_p,$$

respectively, and $\Gamma_{\zeta\psi}$ is such that

$$\Gamma_{\zeta\psi} \left(\Delta_0^{\zeta\psi} \right) = \begin{cases} 0 & \text{for } \Delta_0^{\zeta\psi} = 0 \\ \text{non-zero} & \text{for } \Delta_0^{\zeta\psi} \neq 0. \end{cases}$$

Also, define

$$f_{\partial} = \frac{\delta\tilde{\psi}}{\delta\alpha_p} \Big|_{\partial\mathcal{Z}}, \quad \text{and} \quad e_{\partial} = (-1)^p \frac{\delta\tilde{\psi}}{\delta\alpha_q} \Big|_{\partial\mathcal{Z}}.$$

Then, the restricted contact Hamiltonian vertical vector field $X_{\tilde{h}_{\psi}}|_{\mathcal{A}_{\zeta\psi}^C}$ onto the phase space for distributed-parameter port-Hamiltonian system (see Definition 3.2) gives distributed-parameter port-Hamiltonian systems defined in Definition 2.47. In addition, (34) is satisfied.

Proof. Throughout this proof, $\star\star\alpha = \alpha$ and $\star^{-1}\alpha = \star\alpha$ for any p -form α (see Lemma 2.1), and $\mathcal{A}_{\zeta\psi}^C = \{\tilde{h}_{\psi} = 0\}$ are used. The case of $\{p = 1, q = 3\}$ is proven first and then the case of $\{p = q = 2\}$ is proven.

(Proof for the case of $p = 1, q = 3$): Since

$$\star^{-1} \left[\Delta_p^{\zeta\psi} \wedge (-F_{\psi}^{\zeta p}) \right] = g^{-1} \left(\star\Delta_p^{\zeta\psi}, -F_{\psi}^{\zeta p} \right), \quad \text{and} \quad \star^{-1} \left[\Delta_q^{\zeta\psi} \wedge (-F_{\psi}^{\zeta q}) \right] = \Delta_q^{\zeta\psi} \wedge (-\star F_{\psi}^{\zeta q}),$$

the contact Hamiltonian density function h_ψ is written as

$$\begin{aligned} h_\psi &= g^{-1} \left(\star \Delta_p^{\zeta\psi}, -\mathbf{F}_\psi^{\zeta p} \right) + \Delta_q^{\zeta\psi} \left(-\star \mathbf{F}_\psi^{\zeta q} \right) + \Gamma_{\zeta\psi} \left(\Delta_0^{\zeta\psi} \right), \\ &= \delta^{ab} \left(\star \Delta_p^{\zeta\psi} \right)_a \left(-\mathbf{F}_\psi^{\zeta p} \right)_b + \Delta_q^{\zeta\psi} \left(-\star \mathbf{F}_\psi^{\zeta q} \right) + \Gamma_{\zeta\psi} \left(\Delta_0^{\zeta\psi} \right), \end{aligned}$$

where

$$\star \Delta_p^{\zeta\psi} = \left(\star \Delta_p^{\zeta\psi} \right)_a \sigma^a, \quad \text{and} \quad \mathbf{F}_\psi^{\zeta p} = \left(\mathbf{F}_\psi^{\zeta p} \right)_a \sigma^a.$$

In addition, one has from (37) that

$$\left(\star \Delta_p^{\zeta\psi} \right)_a = (\alpha_p)_a - (\star e_p)_a, \quad \left(\Delta_q^{\zeta\psi} \right) = (\star \alpha_q) - (e_q), \quad \text{where} \quad \star e_p = (\star e_p)_a \sigma^a.$$

The component expression of the restricted contact vertical vector field is obtained from (17) as

$$\begin{aligned} \dot{x}_{(p)}^a \Big|_{\mathcal{A}_{\zeta\psi}^c} &= - \frac{\partial h_\psi}{\partial y_a^{(p)}} \Big|_{\mathcal{A}_{\zeta\psi}^c} = - \delta^{ab} \left(\mathbf{F}_\psi^{\zeta p} \right)_b \Big|_{\mathcal{A}_{\zeta\psi}^c}, \\ \dot{x}_{(q)} \Big|_{\mathcal{A}_{\zeta\psi}^c} &= - \frac{\partial h_\psi}{\partial y^{(q)}} \Big|_{\mathcal{A}_{\zeta\psi}^c} = - \star \mathbf{F}_\psi^{\zeta q} \Big|_{\mathcal{A}_{\zeta\psi}^c}, \\ \dot{y}_a^{(p)} \Big|_{\mathcal{A}_{\zeta\psi}^c} &= \left(\frac{\partial h_\psi}{\partial x_a^{(p)}} + y_a^{(p)} \frac{\partial h_\psi}{\partial z} \right) \Big|_{\mathcal{A}_{\zeta\psi}^c} = - \left(\mathbf{F}_\psi^{\zeta p} \right)_a \Big|_{\mathcal{A}_{\zeta\psi}^c}, \\ \dot{y}^{(q)} \Big|_{\mathcal{A}_{\zeta\psi}^c} &= \left(\frac{\partial h_\psi}{\partial x^{(q)}} + y^{(q)} \frac{\partial h_\psi}{\partial z} \right) \Big|_{\mathcal{A}_{\zeta\psi}^c} = - \star \mathbf{F}_\psi^{\zeta q} \Big|_{\mathcal{A}_{\zeta\psi}^c}, \\ \dot{z} \Big|_{\mathcal{A}_{\zeta\psi}^c} &= \left(h_\psi - y_a^{(p)} \frac{\partial h_\psi}{\partial y_a^{(p)}} - y_a^{(q)} \frac{\partial h_\psi}{\partial y_a^{(q)}} \right) \Big|_{\mathcal{A}_{\zeta\psi}^c} = - \left[g^{-1} \left(\star e_p, \mathbf{F}_\psi^{\zeta p} \right) + e_q \left(\star \mathbf{F}_\psi^{\zeta q} \right) \right] \Big|_{\mathcal{A}_{\zeta\psi}^c}. \end{aligned}$$

These are equivalent to write

$$\begin{aligned} -\frac{\partial \alpha_p}{\partial t} &= \mathbf{F}_\psi^{\zeta p}, \quad -\frac{\partial \alpha_q}{\partial t} = \mathbf{F}_\psi^{\zeta q}, \\ \frac{\partial e_p}{\partial t} &= -\star \mathbf{F}_\psi^{\zeta p} = \frac{\partial}{\partial t} (\star \alpha_p), \quad \frac{\partial e_q}{\partial t} = -\star \mathbf{F}_\psi^{\zeta q} = \frac{\partial}{\partial t} (\star \alpha_q), \quad \text{on } \mathcal{A}_{\zeta\psi}^c, \end{aligned}$$

and

$$\begin{aligned} \dot{z} &= \dot{\psi} = -\star \left(e_p \wedge \mathbf{F}_\psi^{\zeta p} \right) - e_q \left(\star \mathbf{F}_\psi^{\zeta q} \right) = -\star \left[\frac{\delta \tilde{\psi}}{\delta \alpha_p} \wedge d \left(\frac{\delta \tilde{\psi}}{\delta \alpha_q} \right) + \frac{\delta \tilde{\psi}}{\delta \alpha_q} \wedge d \left(\frac{\delta \tilde{\psi}}{\delta \alpha_p} \right) \right] \\ &= -\star d \left(\frac{\delta \tilde{\psi}}{\delta \alpha_p} \wedge \frac{\delta \tilde{\psi}}{\delta \alpha_q} \right) \quad \text{on } \mathcal{A}_{\zeta\psi}^c. \end{aligned}$$

The last equation above yields the following

$$\frac{d\tilde{\psi}}{dt} = \int_{\mathcal{Z}} \dot{\psi} \star 1 = - \int_{\mathcal{Z}} d \left(\frac{\delta \tilde{\psi}}{\delta \alpha_p} \wedge \frac{\delta \tilde{\psi}}{\delta \alpha_q} \right) = - \int_{\partial \mathcal{Z}} \left(\frac{\delta \tilde{\psi}}{\delta \alpha_p} \wedge \frac{\delta \tilde{\psi}}{\delta \alpha_q} \right) = \int_{\partial \mathcal{Z}} \mathbf{f}_\partial \wedge e_\partial = \int_{\partial \mathcal{Z}} e_\partial \wedge \mathbf{f}_\partial,$$

where

$$\mathbf{f}_\partial = \frac{\partial \tilde{\psi}}{\delta \alpha_p} \Big|_{\partial \mathcal{Z}}, \quad \text{and} \quad e_\partial = - \frac{\partial \tilde{\psi}}{\delta \alpha_q} \Big|_{\partial \mathcal{Z}}$$

have been used. Thus, one obtains

$$\frac{d\tilde{\psi}}{dt} = \int_{\mathcal{Z}} \left(\frac{\delta \tilde{\psi}}{\delta \alpha_p} \wedge \frac{\partial \alpha_p}{\partial t} + \frac{\delta \tilde{\psi}}{\delta \alpha_q} \wedge \frac{\partial \alpha_q}{\partial t} \right) = - \int_{\mathcal{Z}} (e_p \wedge \mathbf{f}_p + e_q \wedge \mathbf{f}_q) = \int_{\partial \mathcal{Z}} e_\partial \wedge \mathbf{f}_\partial,$$

from which one has (34).

(Proof for the case of $\{p = q = 2\}$): Since

$$\star^{-1} \left[\left(-\mathbf{F}_\psi^{\zeta p} \right) \wedge \Delta_p^{\zeta \psi} \right] = g^{-1} \left(\Delta_p^{\zeta \psi}, -\star \mathbf{F}_\psi^{\zeta p} \right),$$

the contact Hamiltonian density function h_ψ is written as

$$\begin{aligned} h_\psi &= g^{-1} \left(\Delta_p^{\zeta \psi}, -\star \mathbf{F}_\psi^{\zeta p} \right) + g^{-1} \left(\Delta_q^{\zeta \psi}, -\star \mathbf{F}_\psi^{\zeta q} \right) + \Gamma_{\zeta \psi} \left(\Delta_0^{\zeta \psi} \right) \\ &= \delta^{ab} \left(\Delta_p^{\zeta \psi} \right)_a \left(-\star \mathbf{F}_\psi^{\zeta p} \right)_b + \delta^{ab} \left(\Delta_q^{\zeta \psi} \right)_a \left(-\star \mathbf{F}_\psi^{\zeta q} \right)_b + \Gamma_{\zeta \psi} \left(\Delta_0^{\zeta \psi} \right), \end{aligned}$$

where

$$\Delta_p^{\zeta \psi} = \left(\Delta_p^{\zeta \psi} \right)_a \sigma^a, \quad \Delta_q^{\zeta \psi} = \left(\Delta_q^{\zeta \psi} \right)_a \sigma^a, \quad \star \mathbf{F}_p^{\zeta \psi} = \left(\star \mathbf{F}_p^{\zeta \psi} \right)_a \sigma^a, \quad \star \mathbf{F}_q^{\zeta \psi} = \left(\star \mathbf{F}_q^{\zeta \psi} \right)_a \sigma^a.$$

In addition, one has from (37) that

$$\left(\Delta_p^{\zeta \psi} \right)_a = (\star \alpha_p)_a - (e_p)_a, \quad \left(\Delta_q^{\zeta \psi} \right)_a = (\star \alpha_q)_a - (e_q)_a,$$

where

$$(\star \alpha_p)_a = \frac{1}{2} \epsilon_a^{bc} (\alpha_p)_{bc}, \quad (\star \alpha_q)_a = \frac{1}{2} \epsilon_a^{bc} (\alpha_q)_{bc}.$$

The component expression of the restricted contact vertical vector field is obtained from (17) as

$$\begin{aligned} \dot{x}_{(p)}^a \Big|_{\mathcal{A}_{\zeta \psi}^c} &= - \frac{\partial h_\psi}{\partial y_a^{(p)}} \Big|_{\mathcal{A}_{\zeta \psi}^c} = - \delta^{ab} \left(\star \mathbf{F}_\psi^{\zeta p} \right)_b \Big|_{\mathcal{A}_{\zeta \psi}^c}, \\ \dot{x}_{(q)}^a \Big|_{\mathcal{A}_{\zeta \psi}^c} &= - \frac{\partial h_\psi}{\partial y_a^{(q)}} \Big|_{\mathcal{A}_{\zeta \psi}^c} = - \delta^{ab} \left(\star \mathbf{F}_\psi^{\zeta q} \right)_b \Big|_{\mathcal{A}_{\zeta \psi}^c}, \\ \dot{y}_a^{(p)} \Big|_{\mathcal{A}_{\zeta \psi}^c} &= \left(\frac{\partial h_\psi}{\partial x_a^{(p)}} + y_a^{(p)} \frac{\partial h_\psi}{\partial z} \right) \Big|_{\mathcal{A}_{\zeta \psi}^c} = - \left(\star \mathbf{F}_\psi^{\zeta p} \right)_a \Big|_{\mathcal{A}_{\zeta \psi}^c}, \\ \dot{y}_a^{(q)} \Big|_{\mathcal{A}_{\zeta \psi}^c} &= \left(\frac{\partial h_\psi}{\partial x_a^{(q)}} + y_a^{(q)} \frac{\partial h_\psi}{\partial z} \right) \Big|_{\mathcal{A}_{\zeta \psi}^c} = - \left(\star \mathbf{F}_\psi^{\zeta q} \right)_a \Big|_{\mathcal{A}_{\zeta \psi}^c}, \\ \dot{z} \Big|_{\mathcal{A}_{\zeta \psi}^c} &= \left(h_\psi - y_a^{(p)} \frac{\partial h_\psi}{\partial y_a^{(p)}} - y_a^{(q)} \frac{\partial h_\psi}{\partial y_a^{(q)}} \right) \Big|_{\mathcal{A}_{\zeta \psi}^c} = - \left[g^{-1} \left(e_p, \star \mathbf{F}_\psi^{\zeta p} \right) + g^{-1} \left(e_q, \star \mathbf{F}_\psi^{\zeta q} \right) \right] \Big|_{\mathcal{A}_{\zeta \psi}^c}. \end{aligned}$$

These are equivalent to write

$$\begin{aligned} - \frac{\partial \alpha_p}{\partial t} &= \mathbf{F}_\psi^{\zeta p}, \quad - \frac{\partial \alpha_q}{\partial t} = \mathbf{F}_\psi^{\zeta q}, \\ \frac{\partial e_p}{\partial t} &= - \star \mathbf{F}_\psi^{\zeta p} = \frac{\partial}{\partial t} (\star \alpha_p), \quad \frac{\partial e_q}{\partial t} = - \star \mathbf{F}_\psi^{\zeta q} = \frac{\partial}{\partial t} (\star \alpha_q) \quad \text{on } \mathcal{A}_{\zeta \psi}^c, \end{aligned}$$

and

$$\begin{aligned} \dot{z} &= \dot{\psi} = - \star \left(e_p \wedge \mathbf{F}_\psi^{\zeta p} + e_q \wedge \mathbf{F}_\psi^{\zeta q} \right) = - \star \left[- \frac{\delta \tilde{\psi}}{\delta \alpha_p} \wedge d \left(\frac{\delta \tilde{\psi}}{\delta \alpha_q} \right) + \frac{\delta \tilde{\psi}}{\delta \alpha_q} \wedge d \left(\frac{\delta \tilde{\psi}}{\delta \alpha_p} \right) \right] \\ &= - \star d \left(\frac{\delta \tilde{\psi}}{\delta \alpha_p} \wedge \frac{\delta \tilde{\psi}}{\delta \alpha_q} \right) \quad \text{on } \mathcal{A}_{\zeta \psi}^c. \end{aligned}$$

The last equation above yields the following

$$\frac{d\tilde{\psi}}{dt} = \int_{\mathcal{Z}} \dot{\psi} \star 1 = - \int_{\mathcal{Z}} d \left(\frac{\delta \tilde{\psi}}{\delta \alpha_p} \wedge \frac{\delta \tilde{\psi}}{\delta \alpha_q} \right) = - \int_{\partial \mathcal{Z}} \left(\frac{\delta \tilde{\psi}}{\delta \alpha_p} \wedge \frac{\delta \tilde{\psi}}{\delta \alpha_q} \right) = - \int_{\partial \mathcal{Z}} \mathbf{f}_\partial \wedge e_\partial = \int_{\partial \mathcal{Z}} e_\partial \wedge \mathbf{f}_\partial,$$

where

$$\mathbf{f}_\partial = \frac{\partial \tilde{\psi}}{\delta \boldsymbol{\alpha}_p} \Big|_{\partial \mathcal{Z}}, \quad \text{and} \quad \mathbf{e}_\partial = \frac{\partial \tilde{\psi}}{\delta \boldsymbol{\alpha}_q} \Big|_{\partial \mathcal{Z}}$$

have been used. Thus, one obtains

$$\frac{d\tilde{\psi}}{dt} = - \int_{\mathcal{Z}} (\mathbf{e}_p \wedge \mathbf{f}_p + \mathbf{e}_q \wedge \mathbf{f}_q) = \int_{\partial \mathcal{Z}} \mathbf{e}_\partial \wedge \mathbf{f}_\partial,$$

from which one has (34).

So far the discussion above is for $\mathcal{A}_{\zeta\psi}^{\mathcal{C}}$ and that is valid for a covering U_i containing ζ . Taking into account this, one completes the proof. \square

Remark 3.2. The dimension of the contact manifold \mathcal{C}_ζ over the n -dimensional base space \mathcal{Z} with $\zeta \in \mathcal{Z}$ is given by $\dim \mathcal{C}_\zeta = 2(nC_p + nC_q) + 1$, with $nC_p = n!/(p!(n-p)!)$.

Remark 3.3. As an example of the case of $\{p = q = 2\}$, Maxwell's equations in media without source can be formulated in this approach (see Ref. [17]).

The theorem above is about the case where the dimension of base spaces is $n = 3$. In addition, the quadratic energy functionals given by (35) have only been considered. In what follows, the case of $n = 2$ is stated.

3.2 Two-dimensional Riemannian manifold

For the case $n = 2$, the possible combinations of $0 \leq p, q \leq n$ that satisfy $p + q = n + 1$ are

- $\{p = 1, q = 2\}, \quad \{p = 2, q = 1\}.$

Since the case $\{p = 2, q = 1\}$ can reduce to $\{p = 1, q = 2\}$, attention is concentrated on the case $\{q = 1, p = 2\}$. Taking into account this combination, one has the following theorem.

Theorem 3.2. (*Distributed-parameter port-Hamiltonian system as contact Hamiltonian vertical vector field*): Let (\mathcal{Z}, g) be a 2-dimensional Riemannian manifold, and ψ an energy density function specified later. Fix $\{p = 1, q = 2\}$, let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{Z})$ be a 7-dimensional contact manifold over the base space \mathcal{Z} with $\lambda_{\mathbb{V}} = d_{\mathbb{V}}z - p_a d_{\mathbb{V}}x^a$, $\{\boldsymbol{\sigma}^a\} \in \Gamma\Lambda_{\mathbb{H}}^1\mathcal{K}$ an orthonormal co-frame,

$$\boldsymbol{\alpha}_p = (\boldsymbol{\alpha}_p)_a \boldsymbol{\sigma}^a \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{1,0}\mathcal{K}, \quad \boldsymbol{\alpha}_q = (\boldsymbol{\alpha}_q)_0 \star 1 \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{2,0}\mathcal{K},$$

$$\mathbf{e}_p = (\mathbf{e}_p)_a \boldsymbol{\sigma}^a \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{1,0}\mathcal{K}, \quad \mathbf{e}_q = (\mathbf{e}_q)_0 \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{0,0}\mathcal{K}$$

mixed forms, (x, y, z) canonical coordinates with $x = \{x_{(p)}, x_{(q)}\}$, $y = \{y^{(p)}, y^{(q)}\}$,

$$\begin{aligned} x &= \{(\star \boldsymbol{\alpha}_p)^1, (\star \boldsymbol{\alpha}_p)^2, \star \boldsymbol{\alpha}_q\}, & x_{(p)}^a &= \delta^{ab} (\star \boldsymbol{\alpha}_p)_b, & x_{(q)} &= \star \boldsymbol{\alpha}_q = (\boldsymbol{\alpha}_q)_0, \\ y &= \{(\mathbf{e}_p)_1, (\mathbf{e}_p)_2, \mathbf{e}_q\}. & y_a^{(p)} &= (\mathbf{e}_p)_a, & y_a^{(q)} &= \mathbf{e}_q = (\mathbf{e}_q)_0. \\ z &= \mathcal{E}, \end{aligned}$$

where \mathcal{E} is the value of the energy density such that $\mathcal{E} = \psi$ on $\mathcal{A}_{\zeta\psi}^{\mathcal{C}}$ with ψ depending only on x . Furthermore, let $\tilde{\psi}$ be a quadratic energy functional such that

$$\tilde{\psi} = \frac{1}{2} \int_{\mathcal{Z}} (\boldsymbol{\alpha}_p \wedge \star \boldsymbol{\alpha}_p + \boldsymbol{\alpha}_q \wedge \star \boldsymbol{\alpha}_q) = \int_{\mathcal{Z}} \psi \star 1, \quad \text{so that} \quad \mathbf{e}_p = \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_p} \quad \text{and} \quad \mathbf{e}_q = \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_q}.$$

In addition, choose the contact Hamiltonian functional as

$$\tilde{h}_\psi = \int_{\mathcal{Z}} h_\psi \star 1 = \int_{\mathcal{Z}} \left[\Delta_p^{\zeta\psi} \wedge \mathbf{F}_\psi^{\zeta p} + \Delta_q^{\zeta\psi} \wedge \left(-\mathbf{F}_\psi^{\zeta q} \right) + \Gamma_{\zeta\psi} \left(\Delta_0^{\zeta\psi} \right) \star 1 \right],$$

where $\Delta_0^{\zeta\psi}, \Delta_p^{\zeta\psi}, \Delta_q^{\zeta\psi}$ are defined in (37), $h_\psi \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{0,0}\mathcal{K}$, $\mathbf{F}_\psi^{\zeta p} \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{p,0}\mathcal{K}$, $\mathbf{F}_\psi^{\zeta q} \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{q,0}\mathcal{K}$ are

$$h_\psi = \star^{-1} \left[\Delta_p^{\zeta\psi} \wedge \mathbf{F}_\psi^{\zeta p} \right] + \star^{-1} \left[\Delta_q^{\zeta\psi} \wedge \left(-\mathbf{F}_\psi^{\zeta q} \right) \right] + \Gamma_{\zeta\psi} \left(\Delta_0^{\zeta\psi} \right),$$

$$\mathbf{F}_\psi^{\zeta p} := (-1)^r d \left(\frac{\delta \tilde{\psi}}{\delta \alpha_q} \right) = (-1)^r d \star \alpha_q, \quad \mathbf{F}_\psi^{\zeta q} := d \left(\frac{\delta \tilde{\psi}}{\delta \alpha_p} \right) = d \star \alpha_p,$$

respectively, and $\Gamma_{\zeta\psi}$ is such that

$$\Gamma_{\zeta\psi} \left(\Delta_0^{\zeta\psi} \right) = \begin{cases} 0 & \text{for } \Delta_0^{\zeta\psi} = 0 \\ \text{non-zero} & \text{for } \Delta_0^{\zeta\psi} \neq 0. \end{cases}$$

Also, define

$$\mathbf{f}_\partial = \frac{\delta \tilde{\psi}}{\delta \alpha_p} \Big|_{\partial \mathcal{Z}}, \quad \text{and} \quad \mathbf{e}_\partial = (-1)^p \frac{\delta \tilde{\psi}}{\delta \alpha_q} \Big|_{\partial \mathcal{Z}}.$$

Then, the restricted contact Hamiltonian vertical vector field $X_{\tilde{h}_\psi}|_{\mathcal{A}_{\zeta\psi}^c}$ onto the phase space for distributed-parameter port-Hamiltonian system (see Definition 3.2) gives distributed-parameter port-Hamiltonian systems given in Definition 2.47. In addition, (34) is satisfied.

Proof. One can follow the procedure given in the proof of Theorem 3.1. Throughout this proof, $\star \star \alpha = (-1)^p \alpha$ and $\star^{-1} \alpha = (-1)^p \star \alpha$ for any p -form α (see Lemma 2.2), and $\mathcal{A}_{\zeta\psi}^c = \{ \tilde{h}_\psi = 0 \}$ are used. (Proof for the case of $p = 1, q = 2$): Since

$$\star^{-1} \left[\left(-\mathbf{F}_\psi^{\zeta p} \right) \wedge \Delta_p^{\zeta\psi} \right] = g^{-1} \left(\Delta_p^{\zeta\psi}, -\star \mathbf{F}_\psi^{\zeta p} \right), \quad \text{and} \quad \star^{-1} \left[\Delta_q^{\zeta\psi} \wedge \left(-\mathbf{F}_\psi^{\zeta q} \right) \right] = \Delta_q^{\zeta\psi} \left(-\star \mathbf{F}_\psi^{\zeta q} \right),$$

the contact Hamiltonian density function h_ψ is written as

$$\begin{aligned} h_\psi &= g^{-1} \left(\Delta_p^{\zeta\psi}, -\star \mathbf{F}_\psi^{\zeta p} \right) + \Delta_q^{\zeta\psi} \left(-\star \mathbf{F}_\psi^{\zeta q} \right) + \Gamma_{\zeta\psi} \left(\Delta_0^{\zeta\psi} \right) \\ &= \delta^{ab} \left(\Delta_p^{\zeta\psi} \right)_a \left(-\star \mathbf{F}_\psi^{\zeta p} \right)_b + \left(\Delta_q^{\zeta\psi} \right) \left(-\star \mathbf{F}_\psi^{\zeta q} \right) + \Gamma_{\zeta\psi} \left(\Delta_0^{\zeta\psi} \right), \end{aligned}$$

where

$$\Delta_p^{\zeta\psi} = \left(\Delta_p^{\zeta\psi} \right)_a \sigma^a.$$

In addition, one has from (37) that

$$\left(\Delta_p^{\zeta\psi} \right)_a = (\star \alpha_p)_a - (e_p)_a, \quad \left(\Delta_q^{\zeta\psi} \right) = (\star \alpha_q) - (e_q) = (\alpha_q)_0 - (e_q)_0.$$

The component expression of the restricted contact vertical vector field is obtained from (17) as

$$\begin{aligned} \dot{x}_{(p)}^a \Big|_{\mathcal{A}_{\zeta\psi}^c} &= - \frac{\partial h_\psi}{\partial y_a^{(p)}} \Big|_{\mathcal{A}_{\zeta\psi}^c} = - \delta^{ab} \left(\star \mathbf{F}_\psi^{\zeta p} \right)_b \Big|_{\mathcal{A}_{\zeta\psi}^c}, \\ \dot{x}_{(q)} \Big|_{\mathcal{A}_{\zeta\psi}^c} &= - \frac{\partial h_\psi}{\partial y^{(q)}} \Big|_{\mathcal{A}_{\zeta\psi}^c} = - \star \mathbf{F}_\psi^{\zeta q} \Big|_{\mathcal{A}_{\zeta\psi}^c}, \\ \dot{y}_a^{(p)} \Big|_{\mathcal{A}_{\zeta\psi}^c} &= \left(\frac{\partial h_\psi}{\partial x_a^{(p)}} + y_a^{(p)} \frac{\partial h_\psi}{\partial z} \right) \Big|_{\mathcal{A}_{\zeta\psi}^c} = - \left(\star \mathbf{F}_\psi^{\zeta p} \right)_a \Big|_{\mathcal{A}_{\zeta\psi}^c}, \\ \dot{y}^{(q)} \Big|_{\mathcal{A}_{\zeta\psi}^c} &= \left(\frac{\partial h_\psi}{\partial x^{(q)}} + y^{(q)} \frac{\partial h_\psi}{\partial z} \right) \Big|_{\mathcal{A}_{\zeta\psi}^c} = - \star \mathbf{F}_\psi^{\zeta q} \Big|_{\mathcal{A}_{\zeta\psi}^c}, \\ \dot{z} \Big|_{\mathcal{A}_{\zeta\psi}^c} &= \left(h_\psi - y_a^{(p)} \frac{\partial h_\psi}{\partial y_a^{(p)}} - y^{(q)} \frac{\partial h_\psi}{\partial y^{(q)}} \right) \Big|_{\mathcal{A}_{\zeta\psi}^c} = - \left[g^{-1} \left(e_p, \star \mathbf{F}_\psi^{\zeta p} \right) + e_q \left(\star \mathbf{F}_\psi^{\zeta q} \right) \right] \Big|_{\mathcal{A}_{\zeta\psi}^c}. \end{aligned}$$

These are equivalent to write

$$\begin{aligned} - \frac{\partial \alpha_p}{\partial t} &= \mathbf{F}_\psi^{\zeta p}, \quad - \frac{\partial \alpha_q}{\partial t} = \mathbf{F}_\psi^{\zeta q}, \\ \frac{\partial e_p}{\partial t} &= - \star \mathbf{F}_\psi^{\zeta p} = \frac{\partial}{\partial t} (\star \alpha_p), \quad \frac{\partial e_q}{\partial t} = - \star \mathbf{F}_\psi^{\zeta q} = \frac{\partial}{\partial t} (\star \alpha_q) \quad \text{on } \mathcal{A}_{\zeta\psi}^c, \end{aligned}$$

and

$$\begin{aligned}\dot{z} &= \dot{\psi} = - \star \left(\mathbf{e}_p \wedge \mathbf{F}_\psi^{\zeta p} + \mathbf{e}_q \wedge \mathbf{F}_\psi^{\zeta q} \right) = - \star \left[- \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_p} \wedge d \left(\frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_q} \right) + \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_q} \wedge d \left(\frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_p} \right) \right] \\ &= - \star d \left(\frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_p} \wedge \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_q} \right) \quad \text{on } \mathcal{A}_{\zeta\psi}^{\mathcal{C}}.\end{aligned}$$

The last equation above yields the following

$$\frac{d\tilde{\psi}}{dt} = \int_{\mathcal{Z}} \dot{\psi} \star 1 = - \int_{\mathcal{Z}} d \left(\frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_p} \wedge \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_q} \right) = - \int_{\partial \mathcal{Z}} \left(\frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_p} \wedge \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_q} \right) = \int_{\partial \mathcal{Z}} \mathbf{f}_\partial \wedge \mathbf{e}_\partial = - \int_{\partial \mathcal{Z}} \mathbf{e}_\partial \wedge \mathbf{f}_\partial,$$

where

$$\mathbf{f}_\partial = \left. \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\alpha}_p} \right|_{\partial \mathcal{Z}}, \quad \text{and} \quad \mathbf{e}_\partial = - \left. \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\alpha}_q} \right|_{\partial \mathcal{Z}}$$

have been used. Thus, one obtains

$$\frac{d\tilde{\psi}}{dt} = - \int_{\mathcal{Z}} (\mathbf{e}_p \wedge \mathbf{f}_p + \mathbf{e}_q \wedge \mathbf{f}_q) = \int_{\partial \mathcal{Z}} \mathbf{e}_\partial \wedge \mathbf{f}_\partial,$$

from which one has (34).

So far the discussion above is for $\mathcal{A}_{\zeta\psi}^{\mathcal{C}}$ and that is valid for a covering U_i containing ζ . Taking into account this, one completes the proof. \square

Remark 3.4. The dimension of the contact manifold \mathcal{C}_ζ over the n -dimensional base space \mathcal{Z} with $\zeta \in \mathcal{Z}$ is given by $\dim \mathcal{C}_\zeta = 2({}_nC_p + {}_nC_q) + 1$, with ${}_nC_p = n!/(p!(n-p)!)$.

3.3 One-dimensional Riemannian manifold

For the case of $n = 1$, systems with and without quadratic energy functionals are concerned. In many physical problems, systems on 1-dimensional spatial manifolds are considered as toy models, and sometimes these models are exactly solvable. Thus it is expected that the theorem for the case of $n = 1$ will be useful. For this case, $n = 1$, the possible combination of $0 \leq p, q \leq n$ that satisfy $p + q = n + 1$ is

- $\{p = 1, q = 1\}$.

Taking into account this combination, one has the following theorem.

Theorem 3.3. (*Distributed-parameter port-Hamiltonian system as contact vertical vector field*): Let (\mathcal{Z}, g) be a 1-dimensional Riemannian manifold, and ψ an energy density function specified later. let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{Z})$ be a 5-dimensional contact manifold over the base space \mathcal{Z} with $\lambda_{\mathbb{V}} = d_{\mathbb{V}}z - p_a d_{\mathbb{V}}x^a$, $\sigma \in \Gamma\Lambda_{\mathbb{H}}^1\mathcal{K}$ the orthonormal co-frame,

$$\boldsymbol{\alpha}_p = (\boldsymbol{\alpha}_p)_0 \star 1 \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{1,0}\mathcal{K}, \quad \boldsymbol{\alpha}_q = (\boldsymbol{\alpha}_q)_0 \star 1 \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{1,0}\mathcal{K}, \quad \text{where} \quad \star 1 = \sigma,$$

$$\mathbf{e}_p = (\mathbf{e}_p)_0 \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{0,0}\mathcal{K}, \quad \mathbf{e}_q = (\mathbf{e}_q)_0 \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{0,0}\mathcal{K}$$

mixed forms, (x, y, z) canonical coordinates with $x = \{x_{(p)}, x_{(q)}\}$, $y = \{y^{(p)}, y^{(q)}\}$,

$$\begin{aligned}x &= \{\star \boldsymbol{\alpha}_p, \star \boldsymbol{\alpha}_q\}, & x_{(p)} &= \star \boldsymbol{\alpha}_p = (\boldsymbol{\alpha}_p)_0, & x_{(q)} &= \star \boldsymbol{\alpha}_q = (\boldsymbol{\alpha}_q)_0, \\ y &= \{\mathbf{e}_p, \mathbf{e}_q\}. & y^{(p)} &= (\mathbf{e}_p)_0, & y^{(q)} &= (\mathbf{e}_q)_0. \\ z &= \mathcal{E},\end{aligned}$$

where \mathcal{E} is the value of the energy density such that $\mathcal{E} = \psi$ on $\mathcal{A}_{\zeta\psi}^{\mathcal{C}}$ with ψ depending only on x . In addition let $\tilde{\psi}$ be a functional as

$$\tilde{\psi} = \int_{\mathcal{Z}} \psi((\boldsymbol{\alpha}_p)_0, (\boldsymbol{\alpha}_q)_0) \star 1, \quad \text{so that} \quad \mathbf{e}_p = \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_p}, \quad \text{and} \quad \mathbf{e}_q = \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_q},$$

and the contact Hamiltonian functional as

$$\tilde{h}_\psi = \int_{\mathcal{Z}} h_\psi \star 1 = \int_{\mathcal{Z}} \left[\Delta_p^\zeta \left(-\mathbf{F}_\psi^{\zeta p} \right) + \Delta_q^\zeta \left(-\mathbf{F}_\psi^{\zeta q} \right) + \Gamma_{\zeta\psi} \left(\Delta_0^{\zeta\psi} \right) \star 1 \right],$$

where $\Delta_0^{\zeta\psi}, \Delta_p^{\zeta\psi}, \Delta_q^{\zeta\psi}$ are defined in (37), $h_\psi \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{0,0}\mathcal{K}$, $\mathbf{F}_\psi^{\zeta p} \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{1,0}\mathcal{K}$, $\mathbf{F}_\psi^{\zeta q} \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{1,0}\mathcal{K}$ are

$$h_\psi = \star^{-1} \left[\Delta_p^\zeta \left(-\mathbf{F}_\psi^{\zeta p} \right) + \Delta_q^\zeta \left(-\mathbf{F}_\psi^{\zeta q} \right) \right] + \Gamma_{\zeta\psi} \left(\Delta_0^{\zeta\psi} \right),$$

$$\mathbf{F}_\psi^{\zeta p} := (-1)^r d \left(\frac{\delta\tilde{\psi}}{\delta\alpha_p} \right), \quad \mathbf{F}_\psi^{\zeta q} := d \left(\frac{\delta\tilde{\psi}}{\delta\alpha_q} \right),$$

respectively, and $\Gamma_{\zeta\psi}$ is such that

$$\Gamma_{\zeta\psi} \left(\Delta_0^{\zeta\psi} \right) = \begin{cases} 0 & \text{for } \Delta_0^{\zeta\psi} = 0 \\ \text{non-zero} & \text{for } \Delta_0^{\zeta\psi} \neq 0. \end{cases}$$

Also, define

$$\mathbf{f}_\partial = \frac{\delta\tilde{\psi}}{\delta\alpha_p} \Big|_{\partial\mathcal{Z}}, \quad \text{and} \quad \mathbf{e}_\partial = (-1)^p \frac{\delta\tilde{\psi}}{\delta\alpha_q} \Big|_{\partial\mathcal{Z}}.$$

Then, the restricted contact Hamiltonian vertical vector field $X_{\tilde{h}_\psi}|_{\mathcal{A}_{\zeta\psi}^\kappa}$ onto the phase space for distributed-parameter port-Hamiltonian system (see Definition 3.2) gives distributed-parameter port-Hamiltonian systems given in Definition 2.47. In addition, (34) is satisfied.

Proof. One can follow the procedure given in the proof of Theorem 3.1. Throughout this proof, $\star\star\alpha = \alpha$ and $\star^{-1}\alpha = \star\alpha$ for any p -form α (see Lemma 2.3), and $\mathcal{A}_{\zeta\psi}^c = \{\tilde{h}_\psi = 0\}$ are used. Since

$$\star^{-1} \left[\Delta_p^{\zeta\psi} \left(-\mathbf{F}_\psi^{\zeta p} \right) \right] = \Delta_p^{\zeta\psi} \left(-\star \mathbf{F}_\psi^{\zeta p} \right), \quad \text{and} \quad \star^{-1} \left[\Delta_q^{\zeta\psi} \left(-\mathbf{F}_\psi^{\zeta q} \right) \right] = \Delta_q^{\zeta\psi} \left(-\star \mathbf{F}_\psi^{\zeta q} \right),$$

the contact Hamiltonian density function h_ψ is written as

$$h_\psi = \Delta_p \left(-\star \mathbf{F}_\psi^{\zeta p} \right) + \Delta_q \left(-\star \mathbf{F}_\psi^{\zeta q} \right) + \Gamma_{\zeta\psi} \left(\Delta_0^{\zeta\psi} \right).$$

The component expression of the restricted contact vertical vector field is obtained from (17) as

$$\begin{aligned} \dot{x}^{(p)}|_{\mathcal{A}_{\zeta\psi}^c} &= - \frac{\partial h_\psi}{\partial y^{(p)}} \Big|_{\mathcal{A}_{\zeta\psi}^c} = - \star \mathbf{F}_\psi^{\zeta p} \Big|_{\mathcal{A}_{\zeta\psi}^c}, \\ \dot{x}^{(q)}|_{\mathcal{A}_{\zeta\psi}^c} &= - \frac{\partial h_\psi}{\partial y^{(q)}} \Big|_{\mathcal{A}_{\zeta\psi}^c} = - \star \mathbf{F}_\psi^{\zeta q} \Big|_{\mathcal{A}_{\zeta\psi}^c}, \\ \dot{y}^{(p)}|_{\mathcal{A}_{\zeta\psi}^c} &= \left(\frac{\partial h_\psi}{\partial x^{(p)}} + y^{(p)} \frac{\partial h_\psi}{\partial z} \right) \Big|_{\mathcal{A}_{\zeta\psi}^c} = - \star \mathbf{F}_\psi^{\zeta p} \Big|_{\mathcal{A}_{\zeta\psi}^c}, \\ \dot{y}^{(q)}|_{\mathcal{A}_{\zeta\psi}^c} &= \left(\frac{\partial h_\psi}{\partial x^{(q)}} + y^{(q)} \frac{\partial h_\psi}{\partial z} \right) \Big|_{\mathcal{A}_{\zeta\psi}^c} = - \star \mathbf{F}_\psi^{\zeta q} \Big|_{\mathcal{A}_{\zeta\psi}^c}, \\ \dot{z}|_{\mathcal{A}_{\zeta\psi}^c} &= \left(h_\psi - y^{(p)} \frac{\partial h_\psi}{\partial y^{(p)}} - y^{(q)} \frac{\partial h_\psi}{\partial y^{(q)}} \right) \Big|_{\mathcal{A}_{\zeta\psi}^c} = \left[\mathbf{e}_p \left(-\star \mathbf{F}_\psi^{\zeta p} \right) + \mathbf{e}_q \left(-\star \mathbf{F}_\psi^{\zeta q} \right) \right] \Big|_{\mathcal{A}_{\zeta\psi}^c}. \end{aligned}$$

These are equivalent to write

$$- \frac{\partial \alpha_p}{\partial t} = \mathbf{F}_\psi^{\zeta p}, \quad - \frac{\partial \alpha_q}{\partial t} = \mathbf{F}_\psi^{\zeta q},$$

$$\frac{\partial e_p^0}{\partial t} = \frac{\partial}{\partial t} \frac{\delta\tilde{h}_\psi}{\delta\alpha_p} = - \star \mathbf{F}_\psi^{\zeta p}, \quad \frac{\partial e_q^0}{\partial t} = - \star \mathbf{F}_\psi^{\zeta q} = \frac{\partial}{\partial t} (\star \alpha_q) \quad \text{on } \mathcal{A}_{\zeta\psi}^c,$$

and

$$\begin{aligned}\dot{z} &= \dot{\psi} = -\star \left(\mathbf{e}_p \mathbf{F}_\psi^{\zeta_p} + \mathbf{e}_q \mathbf{F}_\psi^{\zeta_q} \right) = -\star \left[\frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_p} d \left(\frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_q} \right) + \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_q} d \left(\frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_p} \right) \right] \\ &= -\star d \left(\frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_p} \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_q} \right) \quad \text{on } \mathcal{A}_{\zeta\psi}^{\mathcal{C}}.\end{aligned}$$

The last equation above yields the following

$$\frac{d\tilde{\psi}}{dt} = \int_{\mathcal{Z}} \dot{\psi} \star 1 = - \int_{\mathcal{Z}} d \left(\frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_p} \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_q} \right) = - \int_{\partial \mathcal{Z}} \left(\frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_p} \frac{\delta \tilde{\psi}}{\delta \boldsymbol{\alpha}_q} \right) = \int_{\partial \mathcal{Z}} \mathbf{f}_\partial \mathbf{e}_\partial = \int_{\partial \mathcal{Z}} \mathbf{e}_\partial \mathbf{f}_\partial,$$

where

$$\mathbf{f}_\partial = \left. \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\alpha}_p} \right|_{\partial \mathcal{Z}}, \quad \mathbf{e}_\partial = - \left. \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\alpha}_q} \right|_{\partial \mathcal{Z}},$$

have been used. Thus, one obtains

$$\frac{d\tilde{\psi}}{dt} = - \int_{\mathcal{Z}} (\mathbf{e}_p \mathbf{f}_p + \mathbf{e}_q \mathbf{f}_q) = \int_{\partial \mathcal{Z}} \mathbf{e}_\partial \mathbf{f}_\partial,$$

from which one has (34).

So far the discussion above is for $\mathcal{A}_{\zeta\psi}^{\mathcal{C}}$ and that is valid for a covering U_i containing ζ . Taking into account this, one completes the proof. \square

Remark 3.5. The dimension of the contact manifold \mathcal{C}_ζ over the n -dimensional base space \mathcal{Z} with $\zeta \in \mathcal{Z}$ is given by $\dim \mathcal{C}_\zeta = 2({}_nC_p + {}nC_q) + 1$, with ${}_nC_p = n!/(p!(n-p)!)$.

In this section with $n = 1, 2$ and 3 , classes of distributed-parameter port-Hamiltonian systems can be written in terms of restricted contact Hamiltonian vertical vector fields. This contact geometric description gives the following points. Since contact Hamiltonian vertical vector fields are on the Legendre submanifolds, the relations between effort variables $\{\mathbf{e}_p, \mathbf{e}_q\}$ and energy variables $\{\boldsymbol{\alpha}_p, \boldsymbol{\alpha}_q\}$ are preserved. In addition, the values of energy are appropriately chosen along the contact Hamiltonian vertical vector fields on phase space.

4 Information geometry for distributed-parameter port-Hamiltonian systems

It has been shown in Ref. [6] that a contact manifold and a strictly convex function induce a dually flat space that is used in information geometry.

When an energy density functional is strictly convex, one can introduce a dually flat space in a fiber space of a bundle for the distributed-parameter port-Hamiltonian systems. First, one introduces a metric tensor field as follows.

Definition 4.1. (*Fiber metric tensor field for distributed-parameter port-Hamiltonian systems in Stokes-Dirac structure*): Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a $(2\tilde{n}_{pq} + 1)$ -dimensional contact manifold over a base space with $\dim \mathcal{B} = n$ and $\tilde{n}_{pq} = {}nC_p + {}nC_q$ with p, q such that $0 \leq p, q \leq n$ and $p + q = n + 1$, (x, y, z) the canonical coordinates for the fiber space such that $\lambda_{\mathbb{V}} = d_{\mathbb{V}}z - y_a d_{\mathbb{V}}x^a$ with $x = \{x^1, \dots, x^n\}$ and $y = \{y_1, \dots, y_n\}$, and ψ an energy density function (see Definition 2.48). Then the metric tensor field $g^\zeta = g_{ab}^\zeta d_{\mathbb{V}}x^a \otimes d_{\mathbb{V}}x^b$ on $\mathcal{A}_{\zeta\psi}^{\mathcal{C}} (\subset \pi^{-1}(\zeta), \zeta \in \mathcal{B})$ with

$$g_{ab}^\zeta = \frac{\partial^2 \psi}{\partial x^a \partial x^b}, \quad a, b \in \{1, \dots, \tilde{n}_{pq}\} \quad (39)$$

is referred to as the fiber metric tensor field of $\mathcal{A}_{\zeta\psi}^{\mathcal{C}}$ for the contact manifold over the base space.

Given the energy functionals used in Section 3, the corresponding energy density functions and co-energy density functions are calculated as follows.

Proposition 4.1. (*Total Legendre transform of energy density function and co-energy density function*): Let $\tilde{\psi}$ and $\tilde{\varphi}$ be an energy functional and co-energy functionals given in (35) and (36), respectively. In addition, let ψ and φ be the energy density function and the co-energy density function such that

$$\tilde{\psi} = \int_{\mathcal{Z}} \psi \star 1, \quad \text{and} \quad \tilde{\varphi} = \int_{\mathcal{Z}} \varphi \star 1.$$

First, one has the following explicit expressions.

- For the case $n = 3, \{p = 1, q = 3\}$, one has

$$\begin{aligned} \psi(x_{(p)}, x_{(q)}) &= \frac{1}{2} \left[g^{-1}(\star \alpha_p, \star \alpha_p) + (\star \alpha_q)^2 \right] = \frac{1}{2} \left[\left(x_{(p)}^1\right)^2 + \left(x_{(p)}^2\right)^2 + \left(x_{(p)}^3\right)^2 + \left(x_{(q)}\right)^2 \right], \\ \varphi(y^{(p)}, y^{(q)}) &= \frac{1}{2} \left[g^{-1}(\star e_p, \star e_p) + (e_q)^2 \right] = \frac{1}{2} \left[\left(y_1^{(p)}\right)^2 + \left(y_2^{(p)}\right)^2 + \left(y_3^{(p)}\right)^2 + \left(y^{(q)}\right)^2 \right], \end{aligned}$$

where $x_{(p)} = \{x_{(p)}^a\}$, $y^{(p)} = \{y_a^{(p)}\}$ and $x_{(q)}, y^{(q)}$ have been defined in Theorem 3.1.

- For the case $n = 3, \{p = 2, q = 2\}$, one has

$$\begin{aligned} \psi(x_{(p)}, x_{(q)}) &= \frac{1}{2} \left[g^{-1}(\star \alpha_p, \star \alpha_p) + g^{-1}(\star \alpha_q, \star \alpha_q) \right] \\ &= \frac{1}{2} \left[\left(x_{(p)}^1\right)^2 + \left(x_{(p)}^2\right)^2 + \left(x_{(p)}^3\right)^2 + \left(x_{(q)}^1\right)^2 + \left(x_{(q)}^2\right)^2 + \left(x_{(q)}^3\right)^2 \right], \\ \varphi(y^{(p)}, y^{(q)}) &= \frac{1}{2} \left[g^{-1}(e_p, e_p) + g^{-1}(e_q, e_q) \right] \\ &= \frac{1}{2} \left[\left(y_1^{(p)}\right)^2 + \left(y_2^{(p)}\right)^2 + \left(y_3^{(p)}\right)^2 + \left(y_1^{(q)}\right)^2 + \left(y_2^{(q)}\right)^2 + \left(y_3^{(q)}\right)^2 \right], \end{aligned}$$

where $x_{(p)} = \{x_{(p)}^a\}$, $y^{(p)} = \{y_a^{(p)}\}$ and $x_{(q)} = \{x_{(q)}^a\}$, $y^{(q)} = \{y_a^{(q)}\}$ have been defined in Theorem 3.1.

- For the case $n = 2, \{p = 1, q = 2\}$, one has

$$\begin{aligned} \psi(x_{(p)}, x_{(q)}) &= \frac{1}{2} \left[g^{-1}(\star \alpha_p, \star \alpha_p) + (\star \alpha_q)^2 \right] = \frac{1}{2} \left[\left(x_{(p)}^1\right)^2 + \left(x_{(p)}^2\right)^2 + \left(x_{(q)}\right)^2 \right], \\ \varphi(y^{(p)}, y^{(q)}) &= \frac{1}{2} \left[g^{-1}(e_p, e_p) + (e_q)^2 \right] = \frac{1}{2} \left[\left(y_1^{(p)}\right)^2 + \left(y_2^{(p)}\right)^2 + \left(y_{(q)}\right)^2 \right], \end{aligned}$$

where $x_{(p)} = \{x_{(p)}^a\}$, $y^{(p)} = \{y_a^{(p)}\}$ and $x_{(q)}, y^{(q)}$ have been defined in Theorem 3.2.

- For the case $n = 1, \{p = 1, q = 1\}$, one has

$$\begin{aligned} \psi(x_{(p)}, x_{(q)}) &= \frac{1}{2} \left[(\star \alpha_p)^2 + (\star \alpha_q)^2 \right] = \frac{1}{2} \left[\left(x_{(p)}\right)^2 + \left(x_{(q)}\right)^2 \right], \\ \varphi(y^{(p)}, y^{(q)}) &= \frac{1}{2} \left[(e_p)^2 + (e_q)^2 \right] = \frac{1}{2} \left[\left(y^{(p)}\right)^2 + \left(y^{(q)}\right)^2 \right], \end{aligned}$$

where $x_{(p)}, y^{(p)}$ and $x_{(q)}, y^{(q)}$ have been defined in Theorem 3.3.

Then all of the cases above, one has

$$\mathfrak{L}[\psi](y^{(p)}, y^{(q)}) = \varphi(y^{(p)}, y^{(q)}).$$

Proof. One can prove these with straightforward calculations. □

With this Proposition, one can show the following.

Proposition 4.2. *(Components of covariant fiber metric tensor field for distributed-port Hamiltonian systems): The inverse matrix of $\{g_{ab}^\zeta\}$ with $x = \{x_{(p)}, x_{(q)}\}$ in (39) is given as*

$$g_\zeta^{ab} = \frac{\partial^2 \varphi}{\partial y_a \partial y_b}, \quad a, b \in \{1, \dots, \tilde{n}_{pq}\},$$

where $y = \{y^{(p)}, y^{(q)}\}$ and $\tilde{n}_{pq} = {}_n C_p + {}_n C_q$.

Proof. A proof is similar to that found in Ref. [18]. \square

In the standard information geometry summarized in Section 2.6, there are two coordinates referred to as dual coordinates (see Definition 2.41), and analogous coordinates exist for our formulation of systems on contact bundles discussed in Section 3. The following proposition shows that x and y are such analogous coordinates.

Proposition 4.3. *(Dual connections for distributed-parameter port-Hamiltonian systems): With $x^j = \partial \varphi / \partial y_j$, one has*

$$g^\zeta \left(\frac{\partial}{\partial x^b}, \frac{\partial}{\partial y_a} \right) = \delta_b^a, \quad a, b \in \{1, \dots, \tilde{n}_{pq}\},$$

where $\{\partial / \partial x^a\}, \{\partial / \partial y_a\}$ are vertical vectors fields, and $\tilde{n}_{pq} = {}_n C_p + {}_n C_q$.

Proof. It follows from

$$\frac{\partial x^j}{\partial y_a} = \frac{\partial^2 \varphi}{\partial y_a \partial y_j} = g_\zeta^{aj}$$

that

$$g^\zeta \left(\frac{\partial}{\partial x^b}, \frac{\partial}{\partial y_a} \right) = g_{ij}^\zeta \delta_b^i \frac{\partial x^j}{\partial y_a} = g_{ij}^\zeta \delta_b^i g_\zeta^{aj} = \delta_b^a.$$

\square

In the standard information geometry, the dual connections are often discussed (see Definition 2.41). Analogous connections can appear in the present geometry.

Definition 4.2. *(Dual connections on contact bundle): Let $(\mathcal{K}, \lambda_\mathbb{V}, \pi, \mathcal{B})$ a contact bundle, \mathcal{C}_ζ a contact manifold over a point of the base space $\zeta \in \mathcal{B}$, $\mathcal{A}_{\zeta_\psi}^\mathcal{C}$ a Legendre submanifold generated by a function ψ over a base point $\zeta \in \mathcal{B}$, g^ζ the metric tensor field on $\mathcal{A}_{\zeta_\psi}^\mathcal{C}$ (see Definition 4.1), ∇^ζ a connection on the Riemannian manifold $(\mathcal{A}_{\zeta_\psi}^\mathcal{C}, g^\zeta)$, and $X_\mathbb{V}, Y_\mathbb{V}, Z_\mathbb{V}$ vertical vector fields. If another connection $\nabla^{\zeta'}$ satisfies*

$$X_\mathbb{V} [g^\zeta(Y_\mathbb{V}, Z_\mathbb{V})] = g^\zeta(\nabla_{X_\mathbb{V}}^\zeta Y_\mathbb{V}, Z_\mathbb{V}) + g^\zeta(Y_\mathbb{V}, \nabla_{X_\mathbb{V}}^{\zeta'} Z_\mathbb{V}).$$

Then the two connections ∇^ζ and $\nabla^{\zeta'}$ are referred to as dual connections on with respect to g^ζ on the contact bundle.

A realization of connection components of dual connections have been known. In our present case of ψ the following is a trivial identity since ψ is a quadratic function.

Proposition 4.4. *(Component expression of dually flat space on contact bundle): Defining*

$$\Gamma_{abc}^{\zeta(\alpha)} := \frac{1-\alpha}{2} \frac{\partial^3 \psi}{\partial x^a \partial x^b \partial x^c}, \quad \alpha \in \mathbb{R},$$

one has

$$\frac{\partial}{\partial x^a} g_{bc}^\zeta = \Gamma_{abc}^{\zeta(\alpha)} + \Gamma_{acb}^{\zeta(-\alpha)}, \quad a, b \in \{1, \dots, {}_n C_p + {}_n C_q\}.$$

where p and q are given natural numbers such that $0 \leq p, q \leq n$ and $p + q = n + 1$.

Proof. Substituting (39) into the left hand side of the equation above, one completes the proof. \square

Remark 4.1. The dual connections ∇^ζ and $\nabla^{\zeta'}$ with respect to g^ζ are constructed such that

$$\nabla_{\partial/\partial x^a}^\zeta \frac{\partial}{\partial x^b} = \Gamma_{ab}^{\zeta(\alpha)c} \frac{\partial}{\partial x^c}, \quad \nabla_{\partial/\partial x^a}^{\zeta'} \frac{\partial}{\partial x^b} = \Gamma_{ab}^{\zeta(-\alpha)c} \frac{\partial}{\partial x^c},$$

where $\Gamma_{ab}^{\zeta(\alpha)c}$ and $\Gamma_{ab}^{\zeta(-\alpha)c}$ are such that

$$\Gamma_{abc}^{\zeta(\alpha)} = g_{cj}^\zeta \Gamma_{ab}^{\zeta(\alpha)j}, \quad \text{and} \quad \Gamma_{abc}^{\zeta(-\alpha)} = g_{cj}^\zeta \Gamma_{ab}^{\zeta(-\alpha)j}.$$

With discussions above, one finds the following main theorem in this section.

Theorem 4.1. (*Distributed-parameter port-Hamiltonian systems induce dually flat space*): The phase space for distributed-parameter port-Hamiltonian systems \mathcal{A}_ψ^κ (see Definition 3.2) and the quadratic energy functional (35) induce the quadruplet $(\mathcal{A}_{\zeta\psi}^c, g^\zeta, \nabla^\zeta, \nabla^{\zeta'})$.

In accordance with a dually flat space in the standard information geometry (see Definition 2.42), one can introduce such a space in the present geometry as follows.

Definition 4.3. (*Dually flat space in contact bundle*): The quadruplet introduced in Theorem 4.1 is referred to as a dually flat space in a contact bundle.

Analogous to Proposition 2.11, one has the following.

Proposition 4.5. (*Inner products and pairings on a dually flat space in contact bundle*): Let $(\mathcal{A}_{\zeta\psi}^c, g^\zeta, \nabla^\zeta, \nabla^{\zeta'})$ be an \tilde{n} -dimensional dually flat space, $x = \{x^1, \dots, x^{\tilde{n}}\}$ a set of ∇ -affine coordinates, $y = \{y_1, \dots, y_{\tilde{n}}\}$ a set of ∇^* -affine coordinates. If the inner products $T_\xi \mathcal{A}_{\zeta\psi}^c \times T_\xi \mathcal{A}_{\zeta\psi}^c \rightarrow \mathbb{R}$, $(\xi \in \mathcal{A}_{\zeta\psi}^c)$ between the bases $\{\partial/\partial x^1, \dots, \partial/\partial x^{\tilde{n}}\}$ and $\{\partial/\partial y_1, \dots, \partial/\partial y_{\tilde{n}}\}$ are given as

$$\begin{aligned} g^\zeta \left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b} \right) &= g_{ab}^\zeta, & g^\zeta \left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y_b} \right) &= \delta_a^b, \\ g^\zeta \left(\frac{\partial}{\partial y_a}, \frac{\partial}{\partial x^b} \right) &= \delta_b^a, & g^\zeta \left(\frac{\partial}{\partial y_a}, \frac{\partial}{\partial y_b} \right) &= g_{ab}^{\zeta'}, \end{aligned}$$

(i.e., x and y are mutually dual with respect to g^ζ), then one has the following pairings $T_\xi^* \mathcal{A}_{\zeta\psi}^c \times T_\xi \mathcal{A}_{\zeta\psi}^c \rightarrow \mathbb{R}$, $(\xi \in \mathcal{M})$

$$\begin{aligned} dx^a \left(\frac{\partial}{\partial x^b} \right) &= \delta^{ab}, & dx^a \left(\frac{\partial}{\partial y_b} \right) &= g_{ab}^{\zeta'}, \\ dy_a \left(\frac{\partial}{\partial x^b} \right) &= g_{ab}^\zeta, & dy_a \left(\frac{\partial}{\partial y_b} \right) &= \delta_{ab}. \end{aligned}$$

The canonical divergence plays a role in information geometry (see Definition 2.43), and that can be defined in the fiber space as follows.

Definition 4.4. (*Canonical divergence on phase space of distributed-parameter port-Hamiltonian systems*): The function $\mathbb{D}^\zeta : \mathcal{A}_{\zeta\psi}^c \times \mathcal{A}_{\zeta\psi}^c \rightarrow \mathbb{R}$, $(\zeta \in \mathcal{Z})$ such that

$$\mathbb{D}^\zeta (\xi \parallel \xi') := \psi(\xi) + \varphi(\xi') - x^a|_{\xi} y_a|_{\xi'}.$$

is referred to as canonical divergence on phase space of distributed-parameter port-Hamiltonian systems.

The generalized Pythagorean theorem plays a role in the standard information geometry (see Theorem 2.10), and an analogous theorem exists in the present geometry.

Theorem 4.2. (*Generalized Pythagorean theorem for phase space of distributed-parameter port-Hamiltonian systems*): Let ξ', ξ'' and ξ''' be points of $\mathcal{A}_{\zeta\psi}^c$ such that 1. ξ' and ξ'' are connected with the $\nabla^{\zeta'}$ -geodesic curve $\gamma^{\zeta'}$ and 2. ξ'' and ξ''' are connected with the ∇^ζ -geodesic curve γ^ζ . If at the intersection ξ'' the curves γ^ζ and $\gamma^{\zeta'}$ are orthogonal with respect to g^ζ , then one has that

$$\mathbb{D}^\zeta (\xi''' \parallel \xi') = \mathbb{D}^\zeta (\xi''' \parallel \xi'') + \mathbb{D}^\zeta (\xi'' \parallel \xi').$$

Proof. A proof is similar to that found in Ref. [18]. □

5 Concluding remarks

This paper provides a viewpoint that distributed-parameter port-Hamiltonian systems with respect to Stokes-Dirac structures can be written in terms of bundles whose fiber spaces are contact manifolds and base spaces are Riemannian manifolds. Also, it has been shown that one can introduce information geometry for a class of distributed parameter port-Hamiltonian systems.

There are some potential future works that follow from this paper. One is to study the case where energy functionals contains higher order terms, since this paper has only considered the case where energy functionals are quadratic except for 1-dimensional Riemannian manifolds. In addition, the present approach should be extended such that distributed-parameter port-Hamiltonian systems with external sources can be described. Such a class with external sources includes Maxwell's equations with some external sources. We believe that the elucidation of these remaining questions will develop the geometric theories in mathematical sciences and foundation of engineering.

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